Practice Problems

Problem 1 :

Given

$$\forall x\,(P(x) \to Q(x))$$
$$\forall x\,(Q(x) \to R(x))$$
$$\exists x\,\neg R(x)$$

$$\therefore \quad \exists x\,\neg P(x)$$

1. $\forall x\,(P(x) \to Q(x))$
2. $\forall x\,(Q(x) \to R(x))$
3. $\exists x\,\neg R(x)$
4. $\neg R(c)$, for some c — Existential instantiation on 3
5. $Q(c) \to R(c)$, for same c — Universal instantiation on 2
6. $\neg Q(c)$, for same c — Modus Tollens using 4 and 5
7. $P(c) \to Q(c)$, for same c — Universal instantiation on 1
8. $\neg P(c)$, for same c — Modus Tollens on 6 and 7
9. $\exists x\,\neg P(x)$ — Existencdial generalization on 8

Problem 2

Proove $1 + 4 + 7 \dots + (3n-2) = \dfrac{n(3n-1)}{2}$

Solution: For $n=1$, $3n-2 = 3.1-2 = 3-2 = 1$

$$R.H.S = \frac{1(3-1)}{2} = \frac{1 \times 2}{2} = 1$$

So, it is true for $n = 1$

Inductive Step:

Well's assume that the statement is true for $n=k$. So,

$$1+4+7 \dots \dots + (3k-2) = \frac{k(3k-1)}{2}$$

Now, we show that the statement holds for $n=k+1$

That is —

$$1+4+7+ \dots \dots 3(k+1)-2 = \frac{(k+1)(3(k+1)-1)}{2} \quad \dots \dots (1)$$

$$= 1+4+7+ \dots \dots 3k+3-2$$

$$= 1+4+7+ \dots \dots + (3k+1)$$

$$= \underbrace{1+4+7+ \dots \dots +(3k-2)}_{= \frac{k(3k-1)}{2}} + (3k+1)$$

$$= \frac{k(3k-1)}{2} + (3k+1)$$

$$= \frac{k(3k-1) + 2(3k+1)}{2}$$

$$= \frac{3k^2 - k + 6k + 2}{2} = \frac{3k^2 + 5k + 2}{2}$$

$$= \frac{3k^2 + 3k + 2k + 2}{2} = \frac{3k(k+1) + 2(k+1)}{2}$$

$$= \frac{(3k+2)(k+1)}{2} = \frac{(k+1)(3(k+1)-1)}{2}$$

similar to the R.H.S of (1)

Given $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots\cdots + \frac{1}{n(n+1)}$

$$\equiv \sum_{k=1}^{n} \frac{1}{k(k+1)}$$

we do partial fraction of $\frac{1}{k(k+1)}$

Let's consider

$$\frac{1}{k(k+1)} = \frac{A}{k} + \frac{B}{k+1}$$

So, $1 = A(k+1) + B(k)$

For $k=-1$,
$B = -1$
For $k = 0$,
$A = 1$

Thus, $\frac{1}{k(k+1)} = \frac{1}{k} + \frac{-1}{k+1} = \frac{1}{k} - \frac{1}{k+1}$

Therefore.

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k+1}$$

for $k=1$, $\frac{1}{1} - \frac{1}{2}$

$k=2$ $\frac{1}{2} - \frac{1}{3}$

$k=3$ $\frac{1}{3} - \frac{1}{4}$

$\vdots$

$k=n$ $\frac{1}{n} - \frac{1}{n+1}$

Sum $\sum_{k=1}^{n}$ $1 - \frac{1}{n+1} = \frac{n}{n+1}$

So, the equation we obtain is $\frac{n}{n+1}$