PREDICATES AND QUANTIFIERS

Limitations of Propositional Logic:

Consider,

Every CS student at NSU must study discrete mathematics

"Name" is a CS student at NSU

Thus, It looks logical to deduce that

"Name" must study discrete mathematics

↳ Syllogistic method of Aristottle

But,

How do we express it using propositional logic; precisely, using propositional operators?

↳ operators we learnt are not applicable

So, WE NEED NEW TOOL

↳ The PREDICATE LOGIC

Another example:

" Every computer connected to the university n/w is functioning properly "

Given that, MATH3 is a computer in the university n/w we can not conclude that

MATH3 is functioning properly

↓

Limitations ⟶ Motivation / Requirement of New tool.

A few examples

$x > 3$ $\quad x = y + 3$ $\quad$ and $\quad x + y = z$

These statements are neither true or false, as the values of the variables are not specified.

" x is greater than 3 "

has two parts in this statement

1. Variable x, subject of the statement
2. PREDICATE " is greater than 3"

Referes to a property that the subject of statement can have.

Now,

Let's denote " x is greater than 3" by $P(x)$

$P(x)$

is the variable

Predicate " is greater than 3 "

$P(x)$ also means the value of the propositional function P at x.

if x is chosen, $P(x)$ becomes a proposition and it holds a truth value

$P(4) \Rightarrow 4 > 3$, is TRUE

$P(2) \Rightarrow 2 > 3$ is FALSE

Example :

$A(x) \Rightarrow$ denotes the statement

" Computer x is under attack by an intruder "

CSE, MATH — Attacked

BBA, LAW — Not attacked

A (CSE) : TRUE

A (MAT) : TRUE

A (BBA) : FALSE

A (LAW) : FALSE

We can also define statements with more than one variable.

$Q(x,y)$ denotes " $x = y + 3$ "

For $Q(1, 2)$, we obtain $1 = 2 + 3$ FALSE

For $Q(3,0)$, we obtain $3 = 0 + 3$ TRUE

$A(c, n)$ means " Computer "c" is connected to network n "

say,

Computer	N/W
CSE	NSU1
BBA	NSU2

Then. $A(CSE, NSU2) = FALSE$.

Quantifiers

When variables in a propositional fn^c is assigned values, we obtain proposition with certain values.

↓ alternatively,

if we quantify a propositional fn^c, we can get propositions.

↓ Quantification

Quantification expresses the extent to which a predicate is true over a range of elements.

A few examples:

$P(x, y, z)$ $Q(x, y, z)$

$x + y = z$ $x - y = z$

$P(-4, 6, 2)$ is true

$P(5, 2, 10)$ is false

$P(5, x, 7)$ is not a proposition

Now,

$P(1,2,3) \wedge Q(5,4,1)$ is true

$P(1,2,4) \longrightarrow Q(5,4,0)$ is true

| if

$P(1,2,3) \longrightarrow Q(5,4,0)$ is FALSE

⊞ That is, for "some" combinationatos we have the propositions TRUE

for "some", FALSE

for "some", not a proposition even.

Similarly, "all", "some", "many" and "none" these are used in quantifications.

⊞ Two types of quantifications are generally used.

a. Universal quantification

A predicate is true for every element under consideration.

b. Existential quantification

Predicate is true for at least a few or one element

⊞ Domain of Discourse:

If a property is true for all values of a variable in particular domain, called domain of Discourse.

We need universal quantifier.

☐ Similarly, we can have more variables to the propositional fn^c.

For example : $R(x, y, z)$ denoting the statement

$$x + y = z$$

Let's say,

$R(1, 2, 3)$ T
$R(0, 0, 1)$ F } Asks students to fill it up

⋮

In general, statements can involve 'n" variables

$$P(x_1, x_2, x_3 \dots x_n)$$

→ P is called an n-place predicate
→ or, n-ary predicate.

APPLICATION : Consider a statement

if $x > 0$ then $x := x + 1$

$P(x)$ here "$x > 0$", When $P(x)$ is TRUE
x is increased by 1, and
if $P(x)$ is FALSE, assignment statement is not executed.

Definition:

Universal quantification of $P(x)$ is the statement

" $P(x)$ for all values of x in the domain "

Notation : $\forall x \; P(x)$

↳ $\forall$ is the universal quantifier.

$\forall x \; P(x)$: for all $x \, P(x)$
for every $x \, P(x)$

Now,

An element that makes $P(n)$ false is known as counter example of $\forall x \, P(x)$

Example:

Let $P(x)$ be the statement " $x+1 > x$ "

then,

$\forall x \, P(x)$ is TRUE if the domain consists of all real numbers.

Caution: The domain should NOT be empty.

"For all" and "For every"

- all of
- for each
- Given any
- For arbitrary

- "for each"
- "for any"

A statement $\forall x\, P(x)$ is false if and only if $P(x)$ is not always true when x takes value from the domain.

One way to show that $P(x)$ is not always true is

Counter example.

Let $Q(x)$ be the statement " $x < 2$ ",
What is the truth value of quantification $\forall x\, Q(x)$ where the domain consists all real numbers.

FALSE.

Cause as $x \in \mathbb{R}$, x can be 3, and

$3 < 2$.

Another example: $P(x)$ states " $x^2 > 0$ "

so, $\forall x\, P(x)$ T/F ? counter example: $x=0,\ x^2=0$

QUESTION: What is the truth value of $\forall x\, P(x)$, where $P(x)$ is the statement $x^2 < 10$ and the domain consists of positive integers not exceeding 4.

$P(1) \wedge P(2) \wedge P(3) \wedge P(4)$, so $\forall x\, P(x)$ is FALSE

Definition:

Existential Quantifier

Existential quantification of $P(x)$ is the proposition

" There exists an element x in the domain such that $P(x)$ "

Notation: $\exists x \, P(x)$

- Without specifying the domain, statement $\exists x \, P(x)$ has no meaning.
- So, one must define the domain.
- Existential quantification $\exists x P(x)$ is read as

  There's an x such as that $P(x)$

  There's at least one x such that $P(x)$

A few alternatives ... "for some", "for at least one", or "there is

Example:

Let $P(x)$ denote the statement "$x > 3$". for Domain consists of all real numbers.

Then $\exists x \, P(x)$ is true

↳ it is sometimes true

Example:

Let $Q(x)$ denotes statement "$x = x + 1$". What is the truth value of quantification $\exists x\, Q(x)$, where domain consists of all real numbers.

Answer is FALSE

Generally, an implicit assumption is that all domains are non-empty.

AND $\exists x\, Q(x)$ is always FALSE

When all elements in the domain can be listed - say $x_1, x_2, x_3 \dots \dots x_n$, the existential quantification $\exists x\, P(x)$ is the same as

$$P(x_1) \vee P(x_2) \vee P(x_3) \dots P(x_n).$$

example

What is the truth value of $\exists x\ P(x)$, where $P(x)$ is the statement $x^2 > 10$, and the universe discourse consists of positive integers not exceeding 4.

Domain $\{1, 2, 3, 4\}$

The proposition $\exists x\, P(x)$ is the same as

$P(1) \vee P(2) \vee P(3) \vee P(4)$ gives true value

NEGATING QUANTIFIED EXPRESSIONS

Let's say, Domain : All students in this class

We assume,

P(x) denotes

x has taken a course in DM

$\downarrow$ Negation

?

$\forall x P(x)$: Every student in this class has taken DM

$\downarrow$ Negation

It is not the case that every student in this class has taken DM

||| equivalent to

There is at least one student in this class who has not taken a course in M

$\hookrightarrow$ Gives us the existential quantifier.

So, $\exists x(\neg P(x))$

Therefore $\neg \forall x P(x) \equiv \exists x \neg P(x)$

Coments: $\neg \forall x P(x)$ it is true, then $\forall x P(x)$ is FALSE. When $\forall x P(x)$ is False,

Existential Quantification

There exists an element x in the domain such that $P(x)$

We use the notation $\exists x\, P(x)$ for the existential quantification of $P(x)$

$\exists$ is called the existential quantifier

Quantifiers as disjunction

When all elements in the domain can be listed as —

$x_1, x_2, x_3 \ldots \ldots x_n$, the existential quantification $\exists x\, P(x)$ is the same as disjunction

$$P(x_1) \vee P(x_2) \vee P(x_3) \ldots \ldots \vee P(x_n)$$

Comments: Suppose "n objects" in a domain.

To determine whether $\forall x\, P(x)$ is true

we use for loop to see if $P(x)$ is always true

To determine whether $\exists x\, P(x)$ is true

we see if for one "x"

$\exists x\, P(x)$ is true

Other quantifiers:

- In principle, we can define as many quantifiers as we need.

  * " There are exactly two "
  * " There are no more than three "

  A few more ...

  * " There are at least 50 "
  * " There are no less than 7 "

Uniqueness quantifier:

- Denoted as $\exists!$
- $\exists! x P(x)$ that " There exists a unique x such that $P(x)$ is true "

  * There is exactly one
  * There is one and only one

Example:

$\exists! x\ P(x)$, where $P(x)$ denotes $x+1 = 2x$, for $x \in \mathbb{Z}$

Some more examples

① $\forall x<0\ (x^2>0)$ $\quad x \in \mathbb{R}$

② $\exists x>0\ (x^2=2)$ $\quad x \in \mathbb{R}$

Example 1 denotes :

Square of a negative real number is positive

↓ can be written as

$$\forall x\,(x<0 \rightarrow x^2>0)$$

Example 2 denotes :

There exists a real number x with $x>0$ such that $x^2=2$. For instance, $x=\sqrt{2}$

↓ Can be written as

$$\exists x\ (x>0 \wedge x^2=2)$$

So,
- Restriction of universal quantification is the same as quantification of a conditional statement Example 1

- Restriction of existential quantification is the same as the existential quantification of a conjunction. Example 2

Quantifier's Precedence

$\forall$ and $\exists$ have higher precedence than all logical operators.

$$(\forall x\, P(x)) \vee (Q(x))$$

Binding and Free variable

Let's consider the statement $\exists x\,(x+y=1)$.

x → bound, y → free

Here, Quantifier is used on x, but not on y

When a quantifier is used on the variable x it is bound

If a variable is not bound by a quantifier, it is free.

$\forall x\,(P(x) \wedge Q(x))$ and $\forall x\, P(x) \wedge \forall x\, Q(x)$ are logically equivalent

Let's assume $x = a$.

Let's assume $\forall x\,(P(x) \wedge Q(x))$ is true. As $x = a$, so,

$P(a) \wedge Q(a)$ is true

$\Rightarrow$ $P(a)$ true, $Q(a)$ true conjunction operation

Now. As $P(a)$ and $Q(a)$ are both true; for every element in the domain. Thus, we can conclude

$\forall x\, P(x)$ and $\forall x\, Q(x)$ both are true

$\Downarrow$

$\forall x\, P(x) \wedge \forall x\, Q(x)$ true

Similarly, Let's assume that $\forall x P(x) \wedge \forall x Q(x)$ is true

$\Downarrow$ This follows that

$\forall x P(x)$ is true, $\forall x Q(x)$ is true

$\Downarrow$ This implies

if a in the domain, then
$P(a)$, $Q(a)$ are true

$\Downarrow$ this suggests for all "a"

$P(a) \wedge Q(a)$ is true

$\Downarrow$ true for all x

$\forall x \left(P(x) \wedge Q(x) \right)$

Finally, $$\forall x \left(P(x) \wedge Q(x) \right) \equiv \forall x P(x) \wedge \forall x Q(x)$$

Translating English to Logic Using Quantifiers.

" Every student in this class has taken a course in C programming "

Sol^n: When U, the domain, is all students in class
Assume, $C(x)$ denotes " x has taken course in C ".
we can translate by $\forall x\, C(x)$

Sol^n: When U, the domain, is all people.

We need to define a new propositional fn^c.

Let's say that,

S(x) denoting,

"x is a student in this class"

$S(x) \rightarrow C(x)$

if "x is a student in this class, then x has taken course in C

We now can translate

$$\forall x \left(S(x) \rightarrow C(x) \right)$$

Question:

$\forall x \left(S(x) \wedge C(x) \right)$ is not correct.

Why?

Here,

the statement says that

" all people are student in this class and have studied C "

SEE EXAMPLES IN BOOK

Use Predicates & quantifier to express —

Every mail larger than 1 MB will be compressed.

Let's assume $S(m, y)$ denotes

Mail message "m" is larger than y MB.

Then, $y = 1$ MB or other interested value.

$$\forall m \left(S(m, 1) \longrightarrow C(m) \right)$$

where, $C(m)$ denotes

Mail "m" will be compressed

domain "all message"

More examples ...

Premises

" All Lions are fierce "

" Some lions do not drink coffe "

" Some fierce creatures do not drink coffee "

Conclusion

Assume domain consists of all creatures

$P(x)$ denotes: x is a lion

$Q(x)$ denotes: x is fierce (ferocious) → opposite: gentle

$R(x)$ denotes: x drinks coffee

So, $\forall x\,(P(x) \longrightarrow Q(x))$

$\exists x\,(P(x) \wedge \neg R(x)) \equiv \exists x\,(P(x) \longrightarrow \neg R(x))$

We can not . .

$\exists x\,(Q(x) \wedge \neg R(x))$

Consider

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

NESTED QUANTIFIERS

Two quantifiers are nested if one is within the scope of the other.

"Within the scope"

↳ Everything within the scope of a quantifier can be thought of a propositional function.

For example $\forall x\, \exists y\, (x + y = 0)$ → Within the scope

$Q(x)$

↓

$\exists y\, P(x, y)$, where $P(x, y)$ is $x + y = 0$

Example:

- $\forall x\, \forall y\, (x + y = y + x)$ domain: $\mathbb{R}$

  ↓

  says that $x + y = y + x$ for all real numbers

  ↳ This is also known as commutative law

- $\forall x\, \exists y\, (x + y = 0)$ says that

  For every real number x there is a real number y such that $x + y = 0$

- $\forall x\, \forall y\, \forall z\, (x + (y + z) = (x + y) + z)$

  ↳ Associative law

Translate into English

Domain: $\mathbb{R}$

$$\forall x \forall y \left((x>0) \wedge (y<0) \rightarrow (xy<0) \right)$$

For every real number x and
For every real number y
if $x>0$ and $y<0$, then $xy<0$

$x>0$ → Positive
$y<0$ → Negative
xy → Product is negative

Quantification as Loops

If $\forall x \forall y\, P(x,y)$ is true: Loop through for x and for each x, we loop through the values for y.

If $\exists x \forall y\, P(x,y)$ is true: To see whether this nested quantifier is true, we loop through the values for x until we find an x for which $P(x,y)$ is always true when we loop through all values for y.

If $\exists x \exists y\, P(x,y)$ is true: We loop through x until we hit an 'x' for which we find a y that makes $P(x,y)$ true

Example:

$Q(x, y)$ denotes $x+y=0$

Truth values of $\exists y \forall x\ Q(x,y)$?

Domain: real numbers. $\forall x \exists y\ Q(x,y)$?

$\exists y \forall x\ Q(x,y)$:

There is a real number y such that for every real number x $Q(x,y)$

Say, $y = 3$, $x+y=0$ only if $x=-3$

So, not true for other values of "x"

FALSE

$\forall x \exists y\ Q(x,y)$:

For every real number x, there is a real number y such that $Q(x,y)$

Say, $x = 1$	$y = -1$	$x+y=0$	TRUE
$= 2$	$y = -2$	$x+y=0$	
$= 3$	$y = -3$	$x+y=0$	

COMMENTS:

Orders at which quantifiers appear make a difference.

So, $\exists y \forall x\ P(x,y)$ and $\forall x \exists y\ P(x,y)$ are not logically equivalent.

Translate Mathematical statements into Nested Quantifiers.

" Every real number except zero, has a multiplicative inverse "

$\downarrow$ rewrite

For every real number x, if $x \neq 0$, then there exists a real number y such that $xy = 1$.

$\downarrow$

$$\forall x (x \neq 0 \rightarrow \exists y (xy = 1))$$

More example at Book.

Negating Nested Quantifier

Find negation of the statement $\forall x \exists y (xy = 1)$ so that no negation precedes a quantifier.

$$\neg \underbrace{\forall x \exists y (xy = 1)}_{\neg \forall x \, Q(x)} \equiv \exists x \neg (\exists y (xy = 1))$$

$$\equiv \exists x \forall y (\neg (xy = 1))$$

$$\equiv \exists x \forall y (xy \neq 1)$$

This comes from De'Morgan Thr^{m}

$$\neg \exists x \, P(x) \equiv \forall x \neg P(x)$$

$$\neg \forall x \, P(x) \equiv \exists x \neg P(x)$$

Example :

exist → woman

There does not a person who has taken a flight on every airline in the world

$\downarrow$ Negation of

w

There exists a woman who has taken a flight on every airline in the world.

f a

$$\hookrightarrow \exists w \, \forall a \, \exists f \big(P(w, f) \wedge Q(f, a) \big)$$

$P(w, f)$: " w has taken flight f "

$Q(f, a)$: " f is a flight on a "

a : airline

f : flight

w : woman

So, negation of $\exists w \, \forall a \, \exists f \big(P(w, f) \wedge Q(f, a) \big)$ will provide us the answer.

$\hookrightarrow$ Apply De Morgan's Law successively.

$$\neg \exists w \, \forall a \, \exists f \big(P(w, f) \wedge Q(f, a) \big)$$

$$\equiv \forall w \, \neg \forall a \, \exists f \big(P(w, f) \wedge Q(f, a) \big)$$

$$\equiv \forall w \, \exists a \, \neg \exists f \big(P(w, f) \wedge Q(f, a) \big)$$

$$\equiv \forall w \, \exists a \, \forall f \, \neg \big(P(w, f) \wedge Q(f, a) \big)$$

$$\equiv \forall w \, \exists a \, \forall f \big(\neg P(w, f) \vee \neg Q(f, a) \big)$$

So, For every woman, there is an airline such that for all flights, this woman has not taken that flight or that flight is not on this airline.