

Limitations of Propositional Logic:

Consider,

Every CS student of NSU must study discrete mathematics

"Name" is a CS student at NSU

Thus,

It looks logical to deduce that

"Name" must study discrete mathematics

↳ Syllogistic method of Aristotle

But,

How do we express it using propositional logic; precisely, using propositional operators?

↳ operators we learnt are not applicable

So,

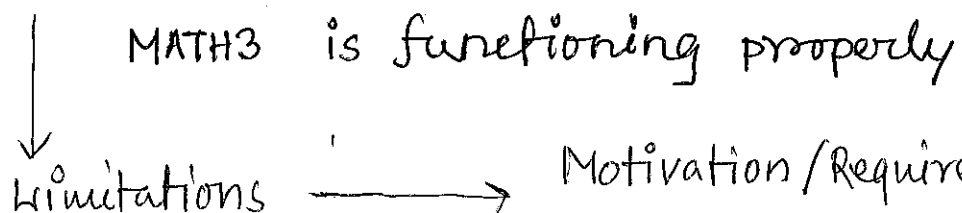
WE NEED NEW TOOL

↳ The PREDICATE LOGIC

Another example:

"Every computer connected to the university n/w is functioning properly"

Given that, MATH3 is a computer in the university n/w we can not conclude that



☐ A few examples

$x > 3$ $x = y + 3$ and $x + y = z$

✓ These statements are neither true or false, as the values of the variables are not specified.

" x is greater than 3 "

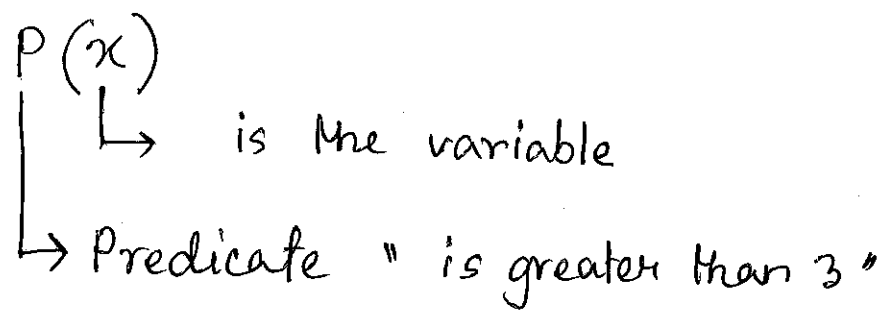
↓ has two parts in this statement

- ① Variable x , subject of the statement
- ② PREDICATE "is greater than 3"

↳ Refers to a property that the subject of statement can have.

Now,

Let's denote " x is greater than 3 " by $P(x)$



✓ $P(x)$ also means the value of the propositional function P at x .

✓ if x is chosen, $P(x)$ becomes a proposition and it holds a truth value

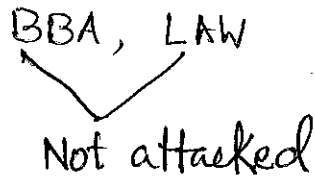
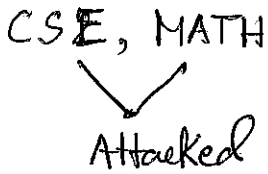
$P(4) \Rightarrow 4 > 3$, is TRUE

$P(2) \Rightarrow 2 > 3$ is FALSE

Example :

$A(x) \Rightarrow$ denotes the statement

" Computer x is under attack by an intruder "



$A(CSE) : TRUE$

$A(BBA) : FALSE$

$A(MAT) : TRUE$

$A(LAW) : FALSE$

We can also define statements with more than one variable.

$Q(x,y)$ denotes " $x = y + 3$ "

For $Q(1, 2)$, we obtain $1 = 2 + 3$ FALSE

For $Q(3, 0)$, we obtain $3 = 0 + 3$ TRUE

$A(c, n)$ means Computer "c" is connected to network "n"

say,	Computer	N/W
	CSE	NSU1
	BBA	NSU2

Then $A(CSE, NSU2) = FALSE$

Quantifiers

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When variables in a propositional fn^c is assigned values, we obtain proposition with certain values.

↓ alternatively,

if we quantify a propositional fn^c, we can get propositions.

↓ Quantification

Quantification expresses the extent to which a predicate is true over a range of elements.

A few examples:

$P(x, y, z)$

$$x + y = z$$

$P(-4, 6, 2)$ is true

$P(5, 2, 10)$ is false

$P(5, x, 7)$ is not a proposition

$Q(x, y, z)$

$$x - y = z$$

$P(1, 2, 3) \wedge Q(5, 4, 1)$ is true

$P(1, 2, 4) \rightarrow Q(5, 4, 0)$ is true

| if

$P(1, 2, 3) \rightarrow Q(5, 4, 0)$ is FALSE

Now,

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☐ That is, for "some" combinations we have the propositions TRUE

for "some", FALSE

for "some", not a proposition even.

Similarly, "all", "some", "many" and "none" these are used in quantifications.

☐ Two types of quantifications are generally used.

a. Universal quantification

A predicate is true for every element under consideration.

b. Existential quantification

Predicate is true for at least a few or one element

☐ Domain of Discourse:

If a property is true for all values of a variable in particular domain, called domain of Discourse.

We need universal quantifier.

Similarly, we can have more variables to the propositional fn^s.

For example :

$R(x, y, z)$ denoting the statement $x + y = z$

Let's say,

$R(1, 2, 3) T$
 $R(0, 0, 1) F$ } Asks students to fill it up

⋮

In general, statements can involve 'n' variables

$P(x_1, x_2, x_3, \dots, x_n)$

↳ P is called an n-place predicate
↳ or, n-ary predicate.

APPLICATION :

Consider a statement
if $x > 0$ then $x := x + 1$

$P(x)$ here " $x > 0$ ", When $P(x)$ is TRUE
 x is increased by 1, and
if $P(x)$ is FALSE, assignment statement is not executed.

Definition:

Universal quantification of $P(x)$ is the statement

" $P(x)$ for all values of x in the domain"

Notation: $\forall x P(x)$

$\rightarrow \forall$ is the universal quantifier.

$\forall x P(x)$:
for all $x P(x)$
for every $x P(x)$

Now,

An element that makes $P(x)$ false is known as counterexample of $\forall x P(x)$

Example:

Let $P(x)$ be the statement " $x+1 > x$ "

then,

$\forall x P(x)$ is TRUE if the domain consists of all real numbers.

NOT

Caution: The domain should be empty.

☐ "For all" and "For every"

→ all of
→ for each
→ Given any
→ For arbitrary

→ "for each"
→ "for any"

⊞ A statement $\forall x P(x)$ is false if and only if $P(x)$ is not always true when x takes value from the domain.

↓ One way to show that $P(x)$ is not always true is counter example.

Let $Q(x)$ be the statement " $x < 2$ ",
 what is the truth value of quantification $\forall x Q(x)$
 where the domain consists all real numbers.

FALSE.

Cause as $x \in \mathbb{R}$, x can be 3, and
 $3 < 2$.

Another example: $P(x)$ states " $x^r > 0$ "

so, $\forall x P(x)$ T/F? counter example:
 $x=0, x^r=0$

QUESTION:

What is the truth value of $\forall x P(x)$, where $P(x)$ is the statement $x^r < 10$ and the domain consists of positive integers not exceeding 4.

$P(1) \wedge P(2) \wedge P(3) \wedge P(4)$, so $\forall x P(x)$ is FALSE

Definition:

Existential Quantifier

Existential quantification of $P(x)$ is the proposition

"There exists an element x in the domain such that $P(x)$ "

Notation: $\exists x P(x)$

✓ Without specifying the domain, statement $\exists x P(x)$ has no meaning.

✓ So, one must define the domain.

✓ Existential quantification, $\exists x P(x)$ is read as

There's an x such as that $P(x)$

There's at least one x such that $P(x)$

A few alternatives ... "for some", "for at least one", or "there is"

Example:

Let $P(x)$ denote the statement " $x > 3$ ". for Domain consists of all real numbers.

Then $\exists x P(x)$ is true

↳ it is sometimes true

Example:

Let $Q(x)$ denotes statement " $x = x + 1$ ". What is the truth value of quantification $\exists x Q(x)$, where domain consists of all real numbers.

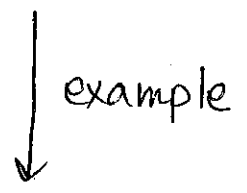
Answer is FALSE

Generally, an implicit assumption is that all domains are non-empty.

AND $\exists x Q(x)$ is always FALSE

When all elements in the domain can be listed - say $x_1, x_2, x_3 \dots \dots x_n$, the existential quantification $\exists x P(x)$ is the same as

$$P(x_1) \vee P(x_2) \vee P(x_3) \dots \dots P(x_n).$$



What is the truth value of $\exists x P(x)$, where $P(x)$ is the statement $x^2 > 10$, and the universe discourse consists of positive integers not exceeding 4.

$$\text{Domain } \{1, 2, 3, 4\}$$

The proposition $\exists x P(x)$ is the same as $P(1) \vee P(2) \vee P(3) \vee P(4)$ gives true value

NEGATING QUANTIFIED EXPRESSIONS

Let's say, Domain: All students in this class

We assume,

$P(x)$ denotes

x has taken a course in M

↓ Negation

?

$\forall x P(x)$: Every student in this class has taken M

↓ Negation

It is not the case that every student in this class has taken M

||| equivalent to

There is at least one student in this class who has not taken a course in M

↳ Gives us the existential quantifier.

So,

$$\exists x (\neg P(x))$$

Therefore

$$\neg \forall x P(x) \equiv \exists x \neg P(x)$$

Comments: $\neg \forall x P(x)$ is true, then $\forall x P(x)$ is FALSE. When $\forall x P(x)$ is false,

Existential Quantification

There exists an element x in the domain such that $P(x)$

We use the notation $\exists x P(x)$ for the existential quantification of $P(x)$

\exists is called the existential quantifier

Quantifiers as disjunction

When all elements in the domain ^{can be} listed as —
 $x_1, x_2, x_3, \dots, x_n$, the existential
 quantification $\exists x P(x)$ is the same as disjunction
 $P(x_1) \vee P(x_2) \vee P(x_3) \dots \vee P(x_n)$

Comments: Suppose "n objects" in a domain.

To determine whether $\forall x P(x)$ is true

we use for loop to see if $P(x)$ is
 always true

To determine whether $\exists x P(x)$ is true

we see if for one "x"

$\exists x P(x)$ is true

Other quantifiers:

W In principle, we can define as many quantifiers as we need.

* "There are exactly two"

* "There are no more than three"

A few more ...

* "There are at least 50"

* "There are no less than 7"

Uniqueness quantifier:

W Denoted as $\exists!$

W $\exists! x P(x)$ that "There exists a unique x such that $P(x)$ is true"

* There is exactly one

* There is one and only one

Example:

$\exists! x P(x)$, where $P(x)$ denotes $x+1 = 2x$, for $x \in \mathbb{Z}$

☐ Some more examples

$$\textcircled{1} \quad \forall x < 0 (x^2 > 0) \quad x \in \mathbb{R}$$

$$\textcircled{2} \quad \exists x > 0 (x^2 = 2) \quad x \in \mathbb{R}$$

Example 1 denotes:

Square of a negative real number is positive

↓ can be written as

$$\forall x (x < 0 \rightarrow x^2 > 0)$$

Example 2 denotes:

There exists a real number x with $x > 0$
such that $x^2 = 2$. For instance, $x = \sqrt{2}$

↓ can be written as

$$\exists x (x > 0 \wedge x^2 = 2)$$

So, \forall Restriction of universal quantification is the same as quantification of a conditional statement Example 1

\exists Restriction of existential quantification is the same as the existential quantification of a conjunction.

Example 2

☐ Quantifier's Precedence

\forall and \exists have higher precedence than all logical operators.

$$(\forall x P(x)) \vee (Q(x))$$

Binding and Free variable

Let's consider the statement $\exists x (x+y=1)$.

Here, quantifier is used on x , but not on y .
 $\exists x$ is labeled "bound" and y is labeled "free".

When a quantifier is used on the variable x it is bound

If a variable is not bound by a quantifier, it's free.

$\forall x (P(x) \wedge Q(x))$ and $\forall x P(x) \wedge \forall x Q(x)$ are logically equivalent

Let's assume $x=a$.

Let's assume $\forall x (P(x) \wedge Q(x))$ is true. As $x=a$, so,

$P(a) \wedge Q(a)$ is true

$\Rightarrow P(a)$ true, $Q(a)$ true conjunction operation

Now. As $P(a)$ and $Q(a)$ are both true; for every element in the domain. Thus, we can conclude

$\forall x P(x)$ and $\forall x Q(x)$ both are true



$\forall x P(x) \wedge \forall x Q(x)$ true

Similarly,

Let's assume that $\forall x P(x) \wedge \forall x Q(x)$ is true

↓ This follows that

$\forall x P(x)$ is true, $\forall x Q(x)$ is true

↓ This implies

if a in the domain, then

$P(a)$, $Q(a)$ are true

↓ this suggests for all "a"

$P(a) \wedge Q(a)$ is true

↓ true for all x

$\forall x (P(x) \wedge Q(x))$

Finally,

$$\forall x (P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$$

☐ Translating English to Logic Using Quantifiers.

"Every student in this class has taken a course in C programming"

Solⁿ: When U , the domain, is all students in class

Assume, $C(x)$ denotes "x has taken course in C".

we can translate by $\forall x C(x)$

Solⁿ: When U , the domain, is all people.

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We need to define a new propositional fn^c.
Let's say that,

$S(x)$ denoting,

" x is a student in this class"

$S(x) \rightarrow C(x)$

\hookrightarrow if " x is a student in this class, then
 x has taken course in C "

We now can translate

$$\forall x (S(x) \rightarrow C(x))$$

Question :

$\forall x (S(x) \wedge C(x))$ is not correct.

Why ?

Here,

the statement says that
"all people are student in this class
and have studied C "

SEE EXAMPLES IN BOOK

Use Predicates & quantifier to express —

Every mail larger than 1 MB will be compressed.

Let's assume $S(m, y)$ denotes Mail message "m" is larger than y MB.

Then, $y = 1 \text{ MB}$ or other interested value.

$$\forall m (S(m, 1) \rightarrow C(m))$$

where, $C(m)$ denotes Mail "m" will be compressed
↳ domain "all message"

More examples ...

- Premises { " All lions are fierce "
 - " Some lions do not drink coffee "
 - " Some fierce creatures do not drink coffee "
- Conclusion

Assume domain consists of all creatures

- $P(x)$ denotes: x is a lion
- $Q(x)$ denotes: x is fierce (ferocious) ↗ opposite: gentle
- $R(x)$ denotes: x drinks coffee

So, $\forall x (P(x) \rightarrow Q(x))$

$\exists x (P(x) \wedge \neg R(x)) \equiv \exists x (P(x) \rightarrow \neg R(x))$
 we cannot ...

$\exists x (Q(x) \wedge \neg R(x))$

consider

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

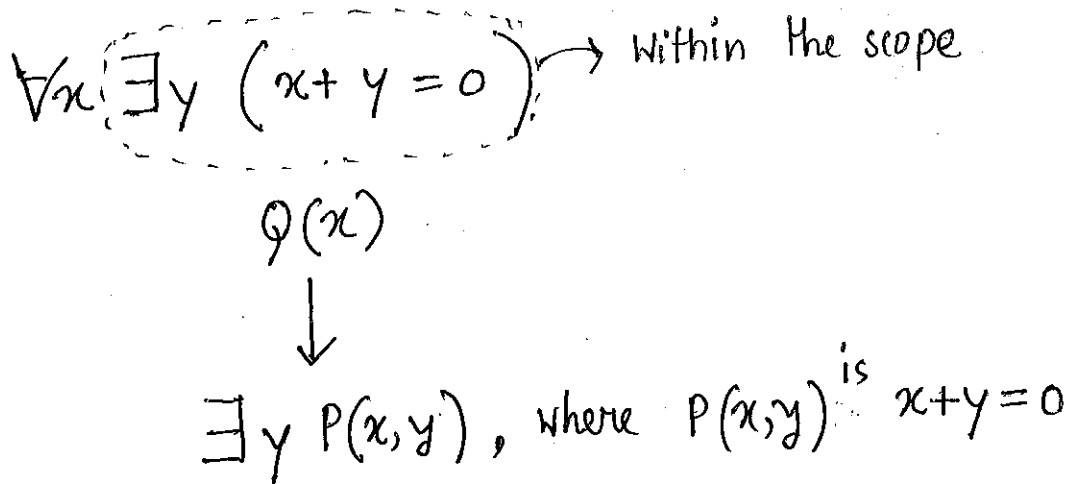
NESTED QUANTIFIERS

Two quantifiers are nested if one is within the scope of the other.

"within the scope"

↳ Everything within the scope of a quantifier can be thought of a propositional function.

For example



Example:

$\forall x \forall y (x+y = y+x)$ domain: \mathbb{R}
 \downarrow
 says that $x+y = y+x$ for all real numbers
 ↳ This is also known as commutative law

$\forall x \exists y (x+y=0)$ says that
 For every real number x there is a real number y such that $x+y=0$

$\forall x \forall y \forall z (x+(y+z) = (x+y)+z)$
 ↳ Associative law

Translate into English

Domain: IR

$$\forall x \forall y ((x > 0) \wedge (y < 0) \rightarrow (xy < 0))$$

For every real number x and

For every real number y

if $x > 0$ and $y < 0$, then $xy < 0$

↓
Positive

↓
Negative

↓
Product is negative

Quantification as Loops

If $\forall x \forall y P(x, y)$ is true: Loop through for x and for each x , we loop through the values for y .

If $\exists x \forall y P(x, y)$ is true: To see whether this nested quantifier is true, we loop through the values for x until we find an x for which $P(x, y)$ is always true when we loop through all values for y .

If $\exists x \exists y P(x, y)$ is true: We loop through x until we hit an " x " for which we find a y that makes $P(x, y)$ true

Example :

$Q(x, y)$ denotes $x+y=0$

Truth values of $\exists y \forall x Q(x, y)$?

Domain : real numbers. $\forall x \exists y Q(x, y)$?

$\exists y \forall x Q(x, y)$:

There is a real number y such that for every real number x $Q(x, y)$

Say, $y=3$, $x+y=0$ only if $x=-3$

So, not true for other values of "x"

FALSE

$\forall x \exists y Q(x, y)$:

For every real number x , there is a real number y such that $Q(x, y)$

Say, $x = 1$	$y = -1$	$x+y=0$	TRUE
$= 2$	$y = -2$	$x+y=0$	
$= 3$	$y = -3$	$x+y=0$	

COMMENTS:

Orders at which quantifiers appear make a difference.

So, $\exists y \forall x P(x, y)$ and $\forall x \exists y P(x, y)$ are not logically equivalent.

Translate Mathematical statements into Nested Quantifiers.

"Every real number except zero, has a multiplicative inverse"

↓ rewrite

For every real number x , if $x \neq 0$, then there exists a real number y such that $xy = 1$.

↓

$$\forall x (x \neq 0 \rightarrow \exists y (xy = 1))$$

More example at Book.

Negating Nested Quantifier

Find negation of the statement $\forall x \exists y (xy = 1)$ so that no negation precedes a quantifier.

$$\begin{aligned} \neg \forall x \exists y (xy = 1) &\equiv \exists x \neg (\exists y (xy = 1)) \\ &\equiv \exists x \forall y (\neg (xy = 1)) \\ &\equiv \exists x \forall y (xy \neq 1) \end{aligned}$$

This comes from De'Morgan Thm

$$\neg \exists x P(x) \equiv \forall x \neg P(x)$$

$$\neg \forall x P(x) \equiv \exists x \neg P(x)$$

Example :

There does not ^{exist} a ^{woman} person who has taken a flight on every ^a airline in the world

Negation of

There exists a ^w woman who has taken a flight on every ^a airline in the world.

→ $\exists w \forall a \exists f (P(w, f) \wedge Q(f, a))$

$P(w, f)$: "w has taken flight f"		a : airline
$Q(f, a)$: "f is a flight on a"		f : flight
		w : woman

So, negation of $\exists w \forall a \exists f (P(w, f) \wedge Q(f, a))$ will provide us the answer.

→ Apply De Morgan's law successively.

$$\neg \exists w \forall a \exists f (P(w, f) \wedge Q(f, a))$$

$$\equiv \forall w \neg \forall a \exists f (P(w, f) \wedge Q(f, a))$$

$$\equiv \forall w \exists a \neg \exists f (P(w, f) \wedge Q(f, a))$$

$$\equiv \forall w \exists a \forall f \neg (P(w, f) \wedge Q(f, a))$$

$$\equiv \forall w \exists a \forall f (\neg P(w, f) \vee \neg Q(f, a))$$

So, For every woman, there is an airline such that for all flights, this woman has not taken that flight or that flight is not on this airline.