RELATIONS

Relations and their Properties

A fundamental way to relate elements of two sets is to form ordered pairs.

Let A and B be sets. A binary relation from A to B is a <u>subset of $A \times B$</u>.

↳ Ordered pairs

A binary relation from A to B is a $\boxed{\text{set } R}$ of ordered pairs where the first element of each ordered pair comes from A
and
second element comes from B.

Notation:

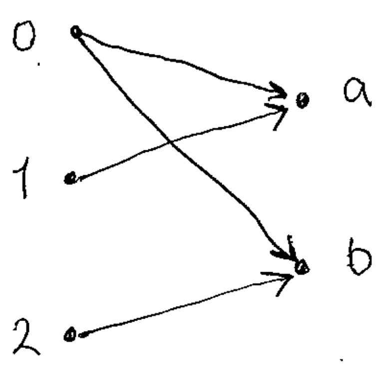
$$a \, R \, b$$
$$\Downarrow$$
$$(a,b) \in R$$
$$\Downarrow$$

a is said to be related to b by R

n-ary relations : Relationships among elements of more than two sets.

Example : Given $A = \{0, 1, 2\}$, $B = \{a, b\}$

Let's consider $\{(0,a), (0,b), (1,a), (2,b)\}$ is a relation from A to B.



Here, 0 to a relation
$$0 \, R \, a$$
and $1 \not R \, b$

Example: Given $A = \{a, b, c\}$, $B = \{1, 2, 3\}$

- If $R = \{(a,1), (b,2), (c,2)\}$, is R a relation from A to B ?

- If $R = \{(1,a), (2,b)\}$, is R a relation from A to B ?

That is,

A binary relation from A to B is a subset of a cartesian product $A \times B$.

$$\Downarrow$$

$$R \subseteq A \times B$$

Representing binary relations

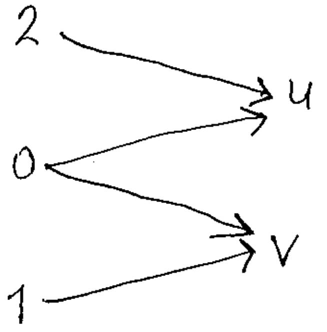
- if $a \, R \, b$, we draw an arrow from a to b

$$a \longrightarrow b$$

Say, $A = \{0, 1, 2\}$, $B = \{u, v\}$

$R = \{(0,u), (0,v), (1,v), (2,u)\}$

Graph :



- We can represent a binary relation

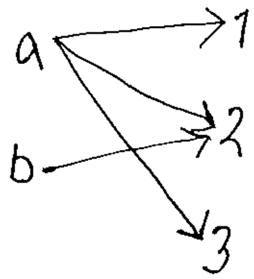
R	u	v
0	x	x
1		x
2	x	

Relations represent one to many relationships between elements in A and B.

Q. What is the difference between a relation and a function from A and B.

Relations :

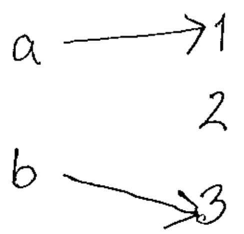
Relations represent one to many relationship between elements in A and B.



Functions :

If a fn^c 'f' is defined on sets A, B, then, $A \rightarrow B$ assigns to each element in the domain set A exactly one element from B.

So, fn^c is a special relation –



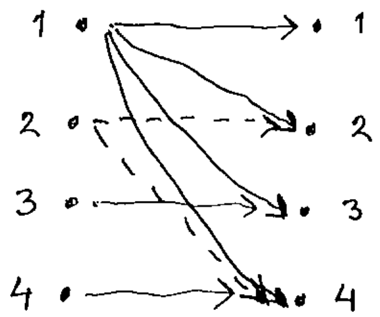
this' a fn^c.

A relation on the set A is a subset of $A \times A$

Example: Given, a set $A = \{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) \mid a \text{ divides } b\}$

$$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$$

We can show the relation graphically as well —



How many relations are there on a set with n elements?

A relation on set "A" is the subset of $A \times A$.

if A has "n" elements, $A \times A$ has n^2 elements.

We know that, any set with "m" elements has 2^m subsets.

For instance if $A = \{1, 2, 3\}$. Total subsets of A is $P(A) = \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}, \{\phi\}\}$

$= 2^{|A|} = 2^3 = 8$

Thus, there will be $2^{n \times n}$ subsets for $A \times A$. Hence, 2^{n^2} relations.

Reflexive Relation: A Relation R on set A is called reflexive if $(a, a) \in R$ for every element $a \in A$.

- Given a set $A = \{1, 2, 3, 4\}$, Let's consider the following relations —

$$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$$
$$R_2 = \{(1,1), (1,2), (2,1)\}$$
$$R_3 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)\}$$

Here, only R_3 is reflexive.

- Given $A = \{1, 2, 3, 4\}$. Is the relation $R = \{(a, b) \mid a \text{ divides } b\}$ reflexive ?

YES; $R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$

- Is the "divides" relation on the set of positive integers reflexive ?

YES; Because a "divides" a (a/a)

- If set of positive integers is replaced by "set of all integers", it is not a relation.

Because, 0 does not divide 0.

- Symmetric and antisymmetric are not opposites. Because, a relation can have both of these properties or may lack both of them.

- Given $A \equiv \{1, 2, 3, 4\}$. Now consider the following relations —

$R_2 = \{(1,1), (1,2), (2,1)\}$ Symmetric

$R_3 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1)$
$(4,4)\}$

Anti-symmetric

$R_4 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$

$R_5 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4)$
$(3,3), (3,4), (4,4)\}$

There's no pair of elements a and b with $a \neq b$ such that (a,b) and (b,a) belong to the relation.

- Is the "divides" relation on the set of positive integers symmetric? or, antisymmetric?

The relation is not symmetric — because, $(1,2)$ is there, but not $(2,1)$

Symmetric Relation :

A relation R on a set A is called symmetric if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$.

Example :

Consider following relations on $\{1, 2, 3, 4\}$

$$R_3 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)\}$$

Symmetric Relation

$$R_2 = \{(1,1), (1,2), (2,1)\}$$ Symmetric Relation

Antisymmetric Relation

A relation R is antisymmetric if for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$. implies that $a = b$

we use quantifier to represent the same ...

$$\forall a \forall b (((a,b) \in R (b,a) \in R) \to (a = b))$$

So, A relation is antisymmetric if and only if there are no pairs of distinct elements a and b with a related to b and b related to a.

Transitive Relations

A relation R on a set A is called transitive if

whenever $(a, b) \in R$ and $(b, c) \in R$,

then $(a, c) \in R$,

for all $a, b, c \in A$.

Use quantifiers to define the same —

$$\forall a \forall b \forall c \, (((a,b) \in R \wedge (b,c) \in R) \rightarrow (a,c) \in R)$$

Example :

Given $A = \{1, 2, 3, 4\}$

$$R_4 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$$

$$R_2 = \{(1,1), (1,2), (2,1)\}$$

↳ Not transitive

↦ $(2,1) \in R$, $(1,2) \in R$ but $(2,2) \notin R$.

Is the "divides" relation on the set of positive integers transitive?

Let's assume a divides b
b divides c

As a divides b, b is a mulliple of "a"
So, $b = ak$, where k is some positive integers.

b divides c, c is a multiple of "b".

So, $c = bl$, where l is some positive integers.

Thus, $c = (ak).l = a(kl)$

again some integers

$\Rightarrow$ c is a multiple of "a"

Therefore, a divides c

Transitive

Combining Relations:

Two relations, say R1 & R2, can be combined in a way two different sets are combined.

Given $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. Relations are

$$R_1 = \{(1,1), (2,2), (3,3)\}$$

$$R_2 = \{(1,1), (1,2), (1,3), (1,4)\}$$

$$R_1 \cap R_2 = \{(1,1)\}$$

$$R_1 - R_2 = \{(2,2), (3,3)\}$$

Composite Relation

Let R is the relation from set A to B
S is the relation from set B to C

Composite of R and S is the relation consisting of ordered pairs (a, c) where, $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$.

We denote composite of R and S by $S \circ R$.

R is a relation from $\{1,2,3\}$ to $\{1,2,3,4\}$
with $R = \{(1,1), (1,4), (2,3), (3,1), (3,4)\}$

S is a relation from $\{1,2,3,4\}$ to $\{0,1,2\}$
with $S = \{(1,0), (2,0), (3,1), (3,2), (4,1)\}$

So, $S \circ R = \{(1,0), (1,1), (2,1), (2,2), (3,0), (3,1)\}$

For instance, a (2,1) is in $S \circ R$.
Because, $(2,3) \in R$

$(3,1) \in S$

b

c

N-ARY RELATIONS

Often relationships require more than 2 sets

- Relationship involving student's name, student's major, and student's grade point average.
- Relationship involving airline, flight number, starting point, destination etc.
- These relations are called N-ary Relations

Study of these relations are important in computer database.

importance: necessary to answer the questions below –

1. Which flight's land at O'Hare Airport between 3 A.M and 4 A.M
2. Which students at your class are sophomores majoring in mathematics and have greater than a 3.0 average.

Definition:

Let $A_1, A_2 \ldots A_n$ be sets. An n-ary relation on these sets in a subset of $A_1 \times A_2 \times \ldots A_n$.

Here, sets $A_1, A_2, \ldots \ldots A_n$ are domains of the relation and n is called the degree of relation.

Example

1. Let R be the relation on $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ consisting of all triples of integers (a, b, c) with $a < b < c$.

Then $(1, 2, 3) \in R$, a relation

$(2, 4, c) \notin R$, not a relation.

Degree : The degree of relation is "3".

Domains : Domains are equal to the set of natural numbers.

2. Let R be the relation $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ consisting of all trip les of integers (a, b, c) in which (a, b, c) form an arithmetic progression.

↓

That is, $(a, b, c) \in R$ iff there is an integer k such that

$$b = a + k$$

$$c = b + k = a + 2k$$

Or,

$$b - a = k, \quad c - b = k$$

So, $(1, 3, 5) \in R$, $(2, 5, 9) \notin R$

Degree : 3

Domains : equal to set of integers

Database and Relations :

- Database, which is a collection of Information, records, often is very large because of our requirement

- We need to add, delete or manupulate records many times per day, and we must minimize the time-needed to do those operations.

- To ensure these operations, various methods are available to represent a database.
  - ↳ Relational data Model is one of them
    - ↳ Based on relations

Database :

- Consists of Records
  - ↳ Which are n-tuples
  - ↳ made up of fields