REPRESENTING RELATIONS

There are many ways to represent relations between finite sets :

i) We can list all the ordered pairs

ii) Tabular Method

Another alternative approach to represent relations to use matrices with entries 0/1 only.

This approach is useful to represent relations in computer programs

Using Matrices for Relations :

Let's assume that R is a relation from

$$A = \{a_1, a_2, a_3 \ldots \ldots a_m\} \quad \text{to}$$

$$B = \{b_1, b_2, b_3 \ldots \ldots b_n\}$$

This relation R can be represented by the matrix $M_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1, & \text{if } (a_i, b_j) \in R \\ 0, & \text{if } (a_i, b_j) \notin R \end{cases}$$

Here, at i, j entry we have "1" if a_i is related to b_j

row and column of Matrix M_R

Example :

Given $A = \{1, 2, 3\}$, $B = \{1, 2\}$

Cardinality $|A| = 3$, $|B| = 2$

So, size of M_R matrices $\underbrace{3}_{\text{row}} \times \underbrace{2}_{\text{column}}$

$$\begin{array}{cc} \text{Column 1} & \text{Column 2} \end{array}$$
$$\begin{bmatrix} \; & \; \\ \; & \; \\ \; & \; \end{bmatrix} \begin{array}{l} \text{row 1} \\ \text{row 2} \\ \text{row 3} \end{array}$$

Given R is a relation from A to B containing (a, b) such that $a \in A$, $b \in B$ and $a > b$

Assume $a_1 = 1$, $a_2 = 2$, $a_3 = 3$
$b_1 = 1$, $b_2 = 2$

So, $R = \{(2,1), (3,1), (3,2)\}$ and the corresponding M_R is :

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

each "1" comes from the ordered pairs with
a being the row
b being the column

Another example

Given, $A = \{1, 2, 3, 4, 5\}$

Let's say the relation R is relation on "A".
$\hookrightarrow A \times A$

Here, $|A| \equiv 5$. So,
$M_R \equiv 5 \times 5$
and is Defined as
$$R = \{(a, b) \mid a \leqslant b\}$$

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Given $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

$R = ?$

$R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), (a_3, b_3), (a_3, b_5)\}$

RELATION TYPES

Reflexive: As we know, a relation R on A is reflexive if $(a, a) \in R$ whenever $a \in A$.

$\downarrow$ That is

R is reflexive if and only if $(a_i, a_i) \in R$ for $i = 1, 2, 3, \ldots \ldots n$

Total no. of element in A

$\downarrow$ Therefore,

$m_{ii} = 1$ for $i = 1, 2, 3 \ldots \ldots n$

each entry in Matrix M_R

This represents elements in the main diagonal

So, M_R for a reflexive relation becomes —

$$M_R = \begin{bmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ 0 & & & & 1 \end{bmatrix}$$

Example :

Given that the relation R on A is represented by

$$M_R = \begin{bmatrix} 1 & \boxed{1} & 0 \\ \boxed{1} & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Antisymmetric ?

Reflexive ?

Symmetric ?

→ (a,b) and $(b,a) \in R$

→ $(a,a), (b,b), (c,c) \in R$

→ As the two entries marked with square are in M_R, the relation is not antisymmetric

Intersection and Union of Relation Matrix

Given, $M_{R1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $M_{R2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

So, $M_{R1 \cup R2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = M_{R1 \vee R2}$

$$M_{R1 \cap R2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = M_{R1 \wedge R2}$$

CLOSURE OF RELATIONS

On many occasion we have relations that are not Reflexive, or aren't symmetric, or are not transitive.

We can add the missing ordered pairs to the relation to make it the given properties.

↓ Resulted relation becomes

Reflexive, symmetric, or transitive or the relation of desired types.

Additional ordered pairs required + the original relation are known as
Reflexive closure
Symmetric closure
Transitive closure

In general,

Let's say a Property X → Reflexive / Symmetric / Transitive

R is a relation that doesn't have all the ordered pairs necessary to hold X Property.

X-closure will be the smallest relation containing R ~~with~~ and the x-property.

↓

X Closure

Reflexive closure

Reflexive closure of a relation R on A would be

$$R \cup \Delta$$

where, $\Delta \equiv$ diagonal relation on A
↳ includes pairs of the form (a,a) with $a \in A$

Say $A = \{1, 2, 3\}$, and the relation R on A is
$R = \{(1,1), (1,2), (2,1), (3,2)\}$

↓ for R to be a reflexive relation $(1,1)$, $(2,2)$, $(3,3)$ should be in R

So, Reflexive closure of R is

$R \cup \Delta$ ↳ here, $\{(2,2), (3,3), (1,1)\}$

$$\equiv \{(1,1), (1,2), (2,1), (3,2)\} \cup \{(2,2), (3,3), (1,1)\}$$

$$\equiv \{(1,1), (1,2), (2,1), (3,2), \underline{\underline{(2,2)}}, \underline{\underline{(3,3)}}\}$$

Added ordered pairs

Symmetric Relation

Relation R on A

For a set $A = \{a_1, a_2 \dots a_n\}$, is symmetric if and only if $(a_j, a_i) \in R$ whenever $(a_i, a_j) \in R$

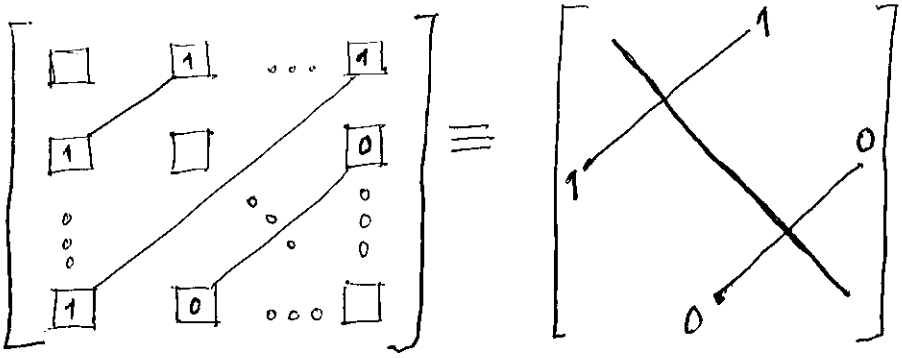
$\downarrow$ That is

$m_{ji} = 1$, whenever $m_{ij} = 1$

This also suggests that —

$m_{ji} = 0$, whenever $m_{ij} = 0$.

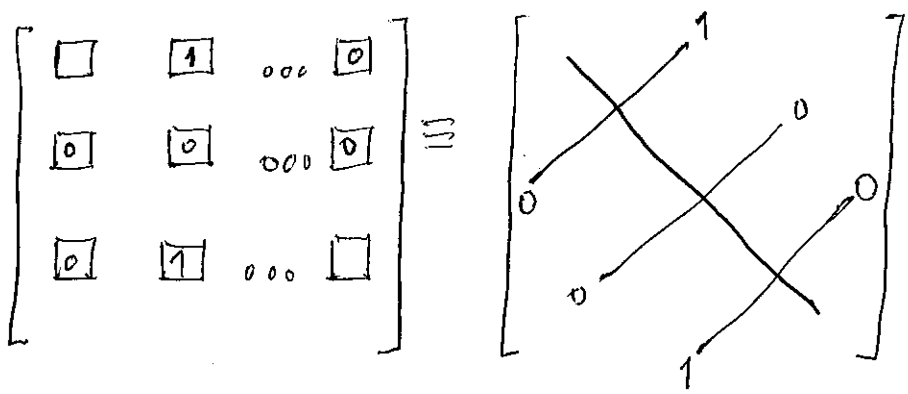
So, $M_R =$



Antisymmetric Relation

Any relation R is antisymmetric if and only if $(a,b) \in R$ and $(b,a) \in R$ imply that $a = b$

$M_R =$



Symmetric closure

Let's assume $A = \{1, 2, 3\}$. We define a relation R on A

$$R = \{(1,1), (1,2), (2, \ ), (2,3), (3,1), (3,2)\}$$

The above relation R is not symmetric. Because—

We don't have $(2,1)$ for $(1,2)$

$(1,3)$ for $(3,1)$

So, we need $(2,1)$ and $(1,3)$ to make R as a symmetric.

↳ (b,a) must be in R, whenever $(a,b) \in R$

Once we add $(2,1)$, $(1,3)$ to R, we obtain the <u>symmetric closure of</u> R

↳ by adding all the ordered pairs of the form (b,a), where $(a,b) \in R$, that are not already present in R.

More generic approach

Symmetric closure of a relation can be constructed by taking the UNION of a relation with its inverse (R^{-1}).
↳ R

So, Symmetric closure $\equiv R \cup R^{-1}$

where $R^{-1} = \{(b,a) \mid (a,b) \in R\}$

Inverse Relation: Let $R \in A \times B$ is a relation from A to B.

Inverse of Relation "R" is denoted by R^{-1}, from B to A.

$$R^{-1} = \{(b,a) \mid (a,b) \in R\}$$

Example: Let's say $R \equiv \{(\underset{a}{1}, \underset{b}{2}), (\underset{a}{2}, \underset{b}{4}), (\underset{a}{3}, \underset{b}{6})\}$ is a relation

so, $R^{-1} \equiv \{(\overset{b}{2}, \overset{a}{1}), (\overset{b}{4}, \overset{a}{2}), (\overset{b}{6}, \overset{a}{3})\}$

TRANSITIVE CLOSURE

By definition, in a transitive relation if (a,b) and (b,c) in Relation R, (a,c) must be in R.

For instance, $R = \{(2,1), (3,1)$