Gregor Cantor's definition on equicardinality

1. Two sets A and B are equicardinal, denoted as $|A| = |B|$, if there's a bijective (also known as one-to-one correspondence) fn^c between A and B (one-to-one Onto)

2. The cardinality of B is greater than or equal to the cardinality of A $(|B| \geqslant |A|)$ if there exists an injective (one-to-one) fn^c from A to B. That is, if

   $f: A \to B$ is an injective fn^c

3. The cardinality of set B is strictly greater than the cardinality of set A if there exists an injective fn^c from A to B, but no bijective fn^c from A to B.

We can use the above finition to define the concept of countablity:

Countably infinite: Any set A is countably infinite if A and $\mathbb{N}$ (set of natural numbers) are equicardinal.

A set is countable if it is either finite or countably infinite. So,

we need a bijection $f: A \to \mathbb{N}$

☐ Show that the set of all integers is countable.

done later

☐ Show that the set of positive rational numbers is countable.

The rational numbers set $\mathbb{Q}$ is defined

$$\mathbb{Q} = \{ p/q \text{, where } p, q \in \mathbb{Z},\ q \neq 0 \}$$

So, the set of positive rational numbers is defined as: $\mathbb{Q}^+ = \{ p/q \text{, where } p, q \in \mathbb{N} \}$

p :

q : column

Draw an array as follows:

	1	2	3	4	5	6	7	
1	$1/1$	$2/1$	$3/1$	$4/1$	$5/1$	$6/1$	$7/1$	...
2	$1/2$	$2/2$	$3/2$	$4/2$	$5/2$	$6/2$	$7/2$	...
3	$1/3$	$2/3$	$3/3$	$4/3$	$5/3$	$6/3$	$7/3$	...
4	$1/4$	$2/4$	$3/4$	$4/4$	$5/4$	$6/4$	$7/4$	...
5	$1/5$	$2/5$	$3/5$	$4/5$	$5/5$	$6/5$	$7/5$	...

So, the initial terms in the list of positive rational numbers are: $1, \frac{1}{2}, 2, 3, \frac{1}{3}, \frac{1}{4}, \frac{2}{3}, 4, 5 \ldots$

$r_1, r_2, r_3, r_4, r_5, r_6, r_7 \ldots \ldots r_n$

positive rational numbers are listed as a sequence.

So, the set of positive rational numbers is countable.

Show that the rationals in [0, 1] is countable

We already have shown that positive rationals set is countable. Here, we consider an interval [0,1] and we have to show that rationals in [0,1] is countable.

how to show

we have to show a bijection from [0, 1] to IN.

Again, we create an array similar to previous proof as follows:

Let's assume that P/q is a rational number with $P, q \in \mathbb{Z}^+$ and $q \neq 0$

Now list out the number as follows:

$q = 1$, and increase q in step of 1 with value of p between $0 \leqslant P \leqslant q$. 1

List out those P/q as obtained using the previous step.

already in list

$$\text{we obtain } \left\{0, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \underbrace{\frac{3}{3}}_{=1}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{5}{5} \cdots\right\}$$

$$\equiv \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \ldots \ldots\right\}$$

As we see, we can clearly have a bijection from [0, 1] to IN.

So, it is countable

Theorem: Assume that I is a countable index set, and A_i is countable as well for every $i \in I$. Then, union-

$$\bigcup_{i \in I} A_i \text{ is countable}$$

The above theorem can be used to prove the already proven statement

" The set of positive rational numbers are countable."

We also can prove if for all the rational numbers. That is:

The set of all Rational numbers is countable.

Proof concept: As we have shown for $[0,1]$, all the rational numbers between $[0,1]$ are countable. That is $\mathbb{Q} \cap [0,1]$ is countable

So,

$\mathbb{Q} \cap [1,2]$ is "

$\mathbb{Q}_i = \mathbb{Q} \cap [i, i+1]$ $\qquad \mathbb{Q} \cap [2,3]$ is "

$\vdots$

We can apply, $\qquad \mathbb{Q} \cap [n, n+1]$ is countable

A countable union of countable sets is countable $\qquad \forall n \in \mathbb{Z}$

$$\bigcup_{i \in \mathbb{Z}} \mathbb{Q}_i = \mathbb{Q}$$

where, each $\mathbb{Q}_i$ is countable

How to check one-to-one concept for any fn^c ?

For any fn^c, check that if $f(a) = f(b)$, then it implies that $a = b$

it ensures the unique images for each element.

Consider, $f(n) = 2n - 1$

so, $f(a) = 2a - 1$

$f(b) = 2b - 1$

Let, $f(a) = f(b)$

That is, $2a - 1 = 2b - 1$

$\Rightarrow 2a = 2b$

$\Rightarrow a = b$

Another example : $f(x) = \dfrac{x-3}{x+2}$

Assume, a, b are elements in the domain.

$$f(a) = \frac{a-3}{a+2}, \quad f(b) = \frac{b-3}{b+2}$$

Let $f(a) = f(b)$, then $\dfrac{a-3}{a+2} = \dfrac{b-3}{b+2}$

$\Rightarrow (a-3)(b+2) = (a+2)(b-3)$

$\Rightarrow \cancel{ab} - 3b + 2a \cancel{-6} = \cancel{ab} + 2b - 3a \cancel{-6}$

$\Rightarrow -3b + 2a = 2b - 3a$

$\Rightarrow 5a = 5b$

so, $a = b$

That is, the fn^c is one-to-one

Another example : $f(x) = 1 - x^2$

Asume that Domain $\mathbb{Z}$ or Co-Domain $\mathbb{R}$

So, $f(a) = f(b)$

$\Rightarrow 1 - a^2 = 1 - b^2$

$\Rightarrow a^2 = b^2$

$\Rightarrow a = \pm b$

that is, if $a = 1$, $b = \pm 1$

so, not one-to-one

◫ Show that the set of integers is countable

$\mathbb{Z}$ = set of integers $= \{0, \pm 1, \pm 2, \pm 3 \dots \dots\}$

As we know, if we can show that $\mathbb{Z}$ can be put into one-to-one correspondence with $\mathbb{N}$, then we can say that $\mathbb{Z}$ is countable.

For instance, one can define the bijection between $\mathbb{Z}$ and $\mathbb{N}$. That is,

$$f: \mathbb{Z} \to \mathbb{N}$$

One way we can put $\mathbb{Z}$ in one-to-one correspondence with $\mathbb{N}$ is the following:

$$\begin{array}{cccccccc} \dots -3, & -2, & -1, & 0, & 1, & 2, & 3, & \dots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 7 & 5 & 3 & 1 & 2 & 4 & 6 & \end{array}$$

Or, simply as follows:

$$\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \dots \dots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 0 & +1 & -1 & +2 & -2 & +3 & -3 & +4 & \end{array}$$