SEQUENCE AND SUMMATIONS

Sequence is a discrete structure to represent an ordered list.

A sequence is a fn^c from a subset of the set of integers to a set S

We use $\{a_n\}$ notation to describe the sequence.

Geometric Progression :

It is of the below form —

$a, ar, ar^2, \ldots \ldots ar^n$

initial term r : common ratio

Sequence $\{b_n\} = (-1)^n$

$1, (-1), 1, (-1) \ldots \ldots$

Arithmetic Progression :

$a, a+d, a+2d, \ldots a+nd$

common difference

initial term

Sequence $\{S_n\} = -1+4n$ is an arithmetic progression

$\{t_n\} = 7-3n$

Geometric Progression

$$\sum_{j=0}^{n} ar^j = a + ar + ar^2 + \dots \dots + ar^n$$
$$= \frac{ar^{n+1} - a}{r-1}, \quad \text{when } r \neq 1$$

Basic Step: $P(0) \Rightarrow ar^0 = a.1 = a$ L.H.S

$$\frac{a.r^{0+1} - a}{r-1} = \frac{ar - a}{r-1}$$

$$\Rightarrow \frac{a(r-1)}{(r-1)} = a$$

Inductive step: Let's consider $P(K)$ is true. So, (inductive hypothesis)

$$\sum_{j=0}^{k} ar^j = a + ar + ar^2 + \dots \dots \dots + ar^k.$$
$$= \frac{ar^{k+1} - a}{r-1} \quad \dots \quad (1)$$

Now, it must be true for $(k+1)$. To show it, we add ar^{k+1} to the both side of (1)

That is – $\displaystyle\sum_{j=0}^{k+1} ar^j = a + ar + ar^2 \dots ar^k + ar^{k+1}$

$$= \frac{ar^{k+1} - a}{r-1} + ar^{k+1}$$

$$= \frac{ar^{k+1} - a + ar^{k+1}.(r-1)}{r-1}$$

$$= \frac{\cancel{ar^{k+1}} - a + ar^{k+2} - \cancel{ar^{k+1}}}{r-1}$$

$$= \frac{ar^{k+2} - a}{r-1} = \frac{ar^{(k+1)+1} - a}{r-1}$$

Proved.

So, if the inductive hypothesis $P(K)$ is true, $P(K+1)$ must be true.

□ Find $\sum_{k=50}^{100} k^2$

We can write $\sum_{k=50}^{100} k^2 = \sum_{k=1}^{100} k^2 - \sum_{k=1}^{49} k^2$

Now, we can use the formula

$$1^2 + 2^2 + 3^2 + 4^2 + \dots \dots \dots + n^2$$
$$= \frac{n(n+1)(2n+1)}{6}$$

Finally, we obtain

$$\sum_{k=50}^{100} k^2 = \frac{100 \times (100+1)(2\times 100+1)}{6} - \frac{49\times(49+1)(2\times 49+1)}{6}$$

$$= \frac{100 \times 101 \times 201}{6} - \frac{49 \times 50 \times 99}{6}$$

$$= 297{,}925$$

□ $\sum_{i=1}^{4} \sum_{j=1}^{3} ij$

$$= \sum_{i=1}^{4} (i + 2i + 3i) = \sum_{i=1}^{4} 6i = 6\sum_{i=1}^{4} i = 6 \times 10$$
$$= 60$$

□ What is the value of $\sum_{s \in \{0,2,4,6,8,10\}} s$

$$= 2 + 4 + 6 + 8 + 10 = 30$$

Cardinality:

Let us assume that S is a set and there are exactly n distinct element in S

↳ Non-negative number

↳ integer

We say that "S" is a finite set

&

n is the cardinality of "S"

↳ Denoted as $|S|$

Example:

Let's define

$$A = \{1, 2, 3, 4, 5\}$$

Cardinality of A is $|A| = 5$

Let's define S is the set of all Bengali alphabets

Cardinality of S is $|S| =$

Observation: As cardinality tells about the size of a set we can use it to compare two different sets

Finite set:

A set S is finite with cardinality $n \in N$ if there is a bijection from the set $\{0, 1, \ldots n-1\}$ to S.

A set is finite if it is not finite

Infinite Set :

A set S is infinite if there exists an injection $f: S \to S$ such that $f(S)$ is a proper subset of S

Example :

$f: \mathbb{N} \to \mathbb{N}$ defined as $f(x) = 3x$

Here, domain is $\mathbb{N}$

Range is obviously the subset of $\mathbb{N}$

So, the set of Natural numbers $\mathbb{N}$ is an infinite set.

Can we measure the size or cardinality of infinite set ?

A set that is either finite or has the same cardinality as the set of positive integers is called countable.

Set of Positive integers : $\mathbb{Z}^+ = \{1, 2, 3, \dots \dots \dots\}$

Show that the set of odd positive integers is a countable set.

Given, $f(n) = 2n-1$ fn^c maps from $\mathbb{Z}^+$ to the set of positive integers.

$\downarrow$ show that

f is one-to-one

f is onto

f is one-to-one

Let's consider that — $f(n) = f(m)$

so, we obtain

$$2n - 1 = 2m - 1$$

$$\Rightarrow \quad n = m$$

That's, similar mapping occur only when n & m are same. This means that $f(n) = 2n-1$ is a one-to-one mapping.

f is onto

Let's assume t is an odd positive integer.

So, t is always 1-less than even integer $2n$.

$n =$	1	2	3	4	5	6	7	8	9	10	11
	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
$2n-1 =$	1	3	5	7	9	11	13	15	17	19	21

Here, bidirectionality demonstrates one-to-one correspondence.

An infinite set is countable if and only if it is possible to list the elements of the set in a sequence.

indexed by positive integers.

Example :

Set of all integers

Set of positive rational numbers.