HW1: Solution

Problem 1

Given

$$f(x) = \frac{1}{x} \qquad f(2) = \frac{1}{2}$$

$$f'(x) = \frac{d}{dx} f(x) = -\frac{1}{x^2} \qquad f'(2) = -\frac{1}{4}$$

$$f''(x) = -(-2)\frac{1}{x^3} = \frac{2}{x^3} \qquad f''(2) = \frac{2}{8} = \frac{1}{4}$$

$$f'''(x) = 2(-3)\cdot\frac{1}{x^4} = -6/x^4 \qquad f'''(2) = -\frac{6}{16} = -\frac{3}{8}$$

We know that

$$f(x) = f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \cdots$$

Taylor's expansion of $f(x)$ centered at $x=a$

$$\Rightarrow \quad f(x) = f(2) + \frac{f'(2)(x-2)}{1!} + \frac{f''(2)(x-2)^2}{2!} + \frac{f'''(2)(x-2)^3}{3!} + \cdots$$

$$= \frac{1}{2} + (x-2)\cdot\left(\frac{-1}{4}\right) + \frac{1}{4}\cdot\frac{1}{2!}(x-2)^2 + \left(-\frac{3}{8}\right)\cdot\frac{(x-2)^3}{3!} + \cdots$$

$$= \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{4}\cdot\frac{(x-2)^2}{2!} - \frac{3}{8}\cdot\frac{(x-2)^3}{3!} + \cdots$$

Problem 2

Given $$f(x) = \frac{\ln(1+x) - x}{x^2}$$

The only part that needs to be expanded here is the term $g(x) = \ln(1+x)$

$g(0) = \ln(1+0) = \ln(1) = 0$	$g'(x) = \frac{1}{1+x}$ $g'(0) = \frac{1}{1} = 1$	$g''(x) = \frac{-1}{(1+x)^2}$ $g''(0) = -1$	$g'''(x) = \frac{2}{(1+x)^3}$ $\Rightarrow g'''(0) = \frac{2}{1} = 2$

So, Taylor's expansion provides —

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \dots$$

$$\Rightarrow \ln(1+x) - x = -\frac{x^2}{2} + \frac{x^3}{3} - \dots \dots$$

$$\Rightarrow \cancel{\frac{\ln(1+x)-x}{x^2} =}$$

$$\Rightarrow \ln(1+x) - x = -\frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$= \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n} x^n$$

if you expand this by assuming $n = 2, 3, 4 \dots$ you obtain $-\frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \dots$

So,

$$\frac{\ln(1+x) - x}{x^2} = \frac{\sum_{n=2}^{\infty} (\;)}{x^2}$$

$$= \frac{1}{x^2} \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n} . x^n$$

$$= \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n} . x^{n-2}$$

$$\equiv \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{(m+2)} x^m$$

Assuming $n = m+2$

Ans

Problem 3

Given $f(x) = e^x$ | $f(0) = 1$

so, $f'(x) = e^x$ | $f'(0) = 1$

$f''(x) = e^x$ | $f''(0) = 1$

$f'''(x) = e^x$ | $f'''(0) = 1$

$\vdots$ | $\vdots$

$f^n(x) = e^x$ | $f^n(0) = 1$

Taylor's expansion provides (for $a=0$)

$$f(x) = f(0) + \frac{f'(0)\cdot x}{1!} + \frac{f''(0)\cdot x^2}{2!} + \frac{f'''(0)}{3!}x^3 + \dots \dots$$

$$\Rightarrow f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \dots \quad \dots (1)$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$$ [if you put $n = 0, 1, 2, 3 \dots \infty$ and add those up, you get (1)]

Upto 3rd term

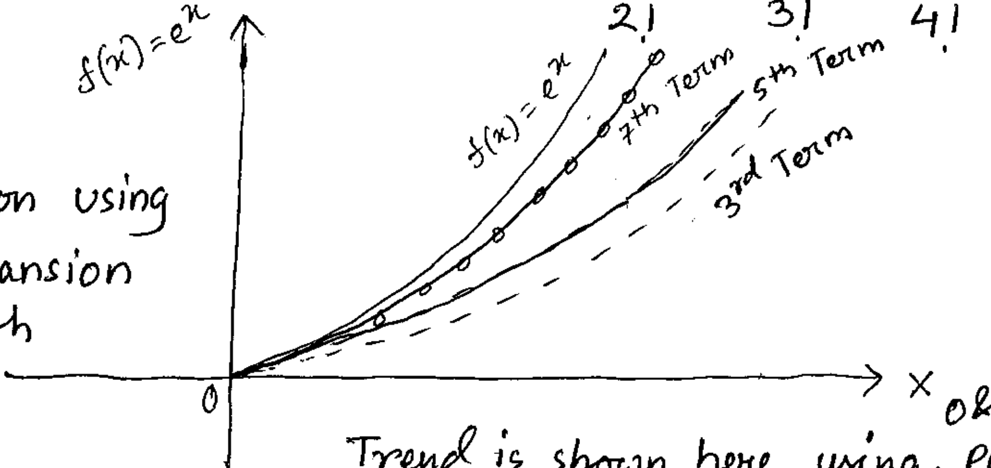
$$f(x) = 1 + x + \frac{x^2}{2!}$$

Upto 5th term

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

Upto 7th term

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!}$$



W Error in approximation using Taylor's expansion reduces with additional terms.

Trend is shown here using positive values of x only.

Problem 4

Given $\frac{x^2}{16} + \frac{y^2}{4} = 1$

By taking implicit derivative, we obtain

That means, y is a fn^c of x

$$\frac{2x}{16} + \frac{2y}{4}\frac{dy}{dx} = 0$$

Here, $\frac{dy}{dx}$ is the slope. So, slope at $(2,-\sqrt{2})$ becomes

$$\frac{2\times 2}{16} + \frac{-2\sqrt{2}}{4}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{1}{4} + \frac{-2\sqrt{2}}{4}\frac{dy}{dx} = 0$$

$$\Rightarrow 1 + 2\sqrt{2}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = +\frac{1}{2\sqrt{2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{2}}$$

So, equation for the tangent line would be:

$$y - (-\sqrt{2}) = \frac{1}{2\sqrt{2}}(x-2)$$

We use the standard eq^n for a line passing through a point (x_1, y_1) with slope m

$$y - \underbrace{y_1}_{(-\sqrt{2})} = \overbrace{m}^{\frac{1}{2\sqrt{2}}}(x - \underbrace{x_1}_{(2)})$$

Problem 5

Given $g(x) = \int_0^x \cos m \, dm$ ...... (1)

We expand "cosm" using Taylor's expansion

$$
\begin{array}{l|l}
f(m) = \cos m & f(0) = 1 \\
f'(m) = -\sin m & f'(0) = 0 \\
f''(m) = -\cos m & f''(0) = -1 \\
f'''(m) = +\sin m & f'''(0) = 0 \\
f^{iv}(m) = \cos m & f^{iv}(0) = 1
\end{array}
$$

So, $f(m) = \cos m = f(0) + \frac{f'(0).m}{1!} + \frac{f''(0).m^2}{2!} + \frac{f'''(0).m^3}{3!} + \frac{f^{iv}(0)}{4!}{}^4 + \dots \dots \dots$

$\Rightarrow f(m) = \cos m = 1 + \frac{m^2}{2!}(-1) + 0 + \frac{m^4}{4!}(1) + \dots \dots \dots$

$\Rightarrow f(m) = \cos m = 1 - \frac{m^2}{2!} + \frac{m^4}{4!} + \dots \dots \dots$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n m^{2n}}{(2n)!} \quad \dots \dots (2)$$

By plugging in (2) in (1),

$$g(x) = \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n m^{2n}}{(2n)!} dm$$

$$\Rightarrow g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Problem 7

Given, $\lim_{x \to 0} \frac{e^x - 1}{x}$

Using Taylor's expansion

$e^x = 1 + x + \frac{1}{2}x^2 e^\xi$

for some ξ betn 0 & x

$\Rightarrow \quad e^x - 1 = x + \frac{1}{2}x^2 e^\xi$

$\Rightarrow \quad \frac{e^x - 1}{x} = \left(\frac{1}{2}x e^\xi + 1\right)$

$\Rightarrow \quad \frac{e^x - 1}{x} - 1 = \frac{1}{2}x e^\xi$

$\Rightarrow \quad \left| \frac{e^x - 1}{x} - 1 \right| = \left| \frac{1}{2}x e^\xi \right|$

$\Rightarrow \quad \left| \frac{e^x - 1}{x} - 1 \right| = \frac{1}{2}|x| e^\xi$

$< |x|$

for all x satisfying $|x| < \ln 2$

When $\xi = 0$

$\frac{1}{2}|x| e^0 = \frac{1}{2}|x|$

$\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

Rate of convergence $O(x)$

when $\xi = x$, $\frac{1}{2}|x| e^{\xi = x} < |x|$

$\Rightarrow \frac{1}{2}e^x < 1$

$\Rightarrow x < \ln 2$

$\Rightarrow |x| < \ln 2$

Problem 8

Given, $\lim_{x \to 0} \frac{\sin x}{x}$

Taylor's expansion provides

$\sin x = x - \frac{x^3}{6}\cos\xi$ for some ξ between 0 & x

$\Rightarrow \quad \sin x - x = -\frac{x^3}{6}\cos\xi$

$\Rightarrow \quad \sin x = x - \frac{x^3}{6}\cos\xi$

$\Rightarrow \quad \frac{\sin x}{x} = 1 - \frac{x^2}{6}\cos\xi$

$$\Rightarrow \quad \frac{\sin x}{x} - 1 = -\frac{x^2}{6}\cos\xi$$

$$\Rightarrow \quad \left|\frac{\sin x}{x} - 1\right| = \left|\frac{-x^2}{6}\cos\xi\right|$$

$$\Rightarrow \quad \left|\frac{\sin x}{x} - 1\right| = \frac{1}{6}\left|x^2\right|\cos\xi \leqslant \frac{1}{6}\left|x^2\right|$$

$$\Rightarrow \quad \lim_{x\to 0}\left|\frac{\sin x}{x}\right| = 1$$ with rate of convergence $O(x^2)$

Problem 6

$$\lim_{n\to\infty}\frac{\sqrt{n^2+1}}{2n+1} = \lim_{n\to\infty}\frac{\sqrt{n^2}}{2n}$$

As $n\to\infty$
$n^2+1 \equiv n$
$2n+1 \equiv 2n$

$$= \lim_{n\to\infty}\frac{n}{2n} = \frac{1}{2}$$

Ans

Problem 9

Self explanatory, and the answer format depends on student's ability; open book question.

Problem 10

You can use multiple approach to solve this problem.

		Remainder
$\frac{1234}{2}$	617	0
$\frac{617}{2}$	308	1
$\frac{308}{2}$	154	0
$\frac{154}{2}$	77	0
$\frac{77}{2}$	38	1
$\frac{38}{2}$	19	0
$\frac{19}{2}$	9	1
$\frac{9}{2}$	4	1
$4/2$	2	0
$2/2$	1	0
$1/2$	0	1

Precedence/ MSB

	integer
$0.125 \times 2 = 0.25$	0
$0.25 \times 2 = 0.5$	0
$0.5 \times 2 = 1$	1

So, $(1234.125)_{10}$

$= (10011010010.001)_2$

Ans