Problem 3

As the fixed-point iteration scheme converges to the limit z, we can write

Hints: Think about the definition of fixed point $x = f(x)$ ↳ fixed point. As we see a convergence here, we can consider z as the fixed point.

$$z = 2z - Mz^2$$
$$\Rightarrow Mz^2 = z$$
$$\Rightarrow Mz = 1$$
$$\Rightarrow z = \frac{1}{A}$$ answer for part a

For the convergence condition is $|g'(x)| < 1$

So, $g'(x) = f'(x)$ [as we consider $f(x)$ as the fn^c]

$$\Rightarrow f'(x) = 2 - 2Mx$$

Thus,

$$|2 - 2Mx| < 1$$

$$\Rightarrow 2 - 2Mx < 1$$
$$\Rightarrow -2Mx < -1$$
$$\Rightarrow 2Mx > 1$$
$$\Rightarrow x > \frac{1}{2M}$$

$$-(2 - 2Mx) < 1$$
$$\Rightarrow 2 - 2Mx > -1$$
$$\Rightarrow 3 > 2Mx$$
$$\Rightarrow 2Mx < 3$$
$$\Rightarrow x < \frac{3}{2M}$$

(b) For $x_0 = \frac{\pi}{2}$, formula for third-order Taylor expansion is as follows

$$P_3(x) = f(x_0) + (x-x_0)f'(x_0) + \frac{(x-x_0)^2 f''(x_0)}{2!} + \frac{(x-x_0)^3 f'''(x_0)}{3!} \quad \ldots \ldots (1)$$

$f(x) = \sin x \Rightarrow f(x_0) = 1$

$f'(x) = \cos x \Rightarrow f'(\pi/2) = \cos \pi/2 = 0$

$f''(x) = -\sin x \Rightarrow f''(\pi/2) = -1$

$f'''(x) = -\cos x \Rightarrow f'''(\pi/2) = 0$

as this term is zero

Using the derivative values:

$$P_3(x) = 1 \cancel{0} + (x-\pi/2)\,0 + \frac{(x-\pi/2)^2 \cdot (-1)}{2!} + 0$$

$\cancel{\Rightarrow P_3(x) = (x-\pi/2) + \frac{1}{2}}$

$\Rightarrow P_3(x) = 1 + (x-\pi/2)^2$

Problem 2 Convergence of a fixed-point iteration scheme requires $|g'(\xi)| < 1$ within the interval

Here, convergence is occurring at $\sqrt{P} = 2$. (Fixed Point)

Thus, i) $g_1'(x) = A_1'(x) = \frac{d}{dx}\, 4 + x - x^2 = -2x + 1$

$\Rightarrow g_1'(2) = -2 \times 2 + 1 = -3$

So, $|g_1'(2)| = |A_1'(2)| = 3$, which is greater than 1

Therefore, this diverges.

Problem 2

(b) Given $f(x) = x^2 - p$ | Newton's method

$\Rightarrow f'(x) = 2x$ | $x_{k+1} = x_k - \dfrac{f(x_k)}{f'(x_k)}$

Here, $p = 4$, $f(x) = x^2 - 4$.

Therefore

$$
\begin{aligned}
x_{k+1} &= x_k - \frac{x_k^2 - 4}{2x_k} \\
&= \frac{2x_k^2 - x_k^2 + 4}{2x_k} \\
&= \frac{x_k^2 + 4}{2x_k} = \frac{x_k^2}{2x_k} + \frac{4}{2x_k} \\
&= \frac{x_k}{2} + \frac{2}{x_k}
\end{aligned}
$$

Answer

According to the formula we derived in class, we can write

$$
\begin{aligned}
W_0(x) &= \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} \\
&= \frac{(x-1)(x-2)(x-3)}{(-1)(-2)(-3)} = \cancel{-\frac{1}{6} f} \\
&= -\frac{1}{6}(x-1)(x-2)(x-3)
\end{aligned}
$$

Problem 5:

Let us create the difference table (Forward) using the points given

We know

$$\Delta f_i = f_{i+1} - f_i$$

$$\Delta^2 f_i = \Delta f_{i+1} - \Delta f_i$$

$$= (f_{i+2} - f_{i+1}) - (f_{i+1} - f_i)$$

$$= f_{i+2} - f_{i+1} - f_{i+1} + f_i$$

$$= f_{i+2} - 2f_{i+1} + f_i$$

So,

$$\Delta f_0 = f_1 - f_0$$

$$\Delta^2 f_0 = f_2 - 2f_1 + f_0$$

$$\Delta^3 f_0 = f_3 + 3f_1 - 3f_2 - f_0$$

$$\Delta^3 f_i = \Delta^2 f_{i+1} - \Delta^2 f_i$$

$$= f_{i+1+2} - 2f_{i+1+1} + f_{i+1} - (f_{i+2} - 2f_{i+1} + f_i)$$

$$\Delta^3 f_i = f_{i+3} + 3f_{i+1} - 3f_{i+2} - f_i$$

Using these formula, we obtain the below table:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	-3				
		6			
1	3		2		
		8		6	
2	11		8		0
		16		6	
3	27		14		0
		30		6	
4	57		20		
		50			
5	107				

Problem 4

We need to revise the theory of fixed point. I hope this helps ~~to~~ you to understand the content.

However, it's not mandatory to revise the theory in order to solve the problem.

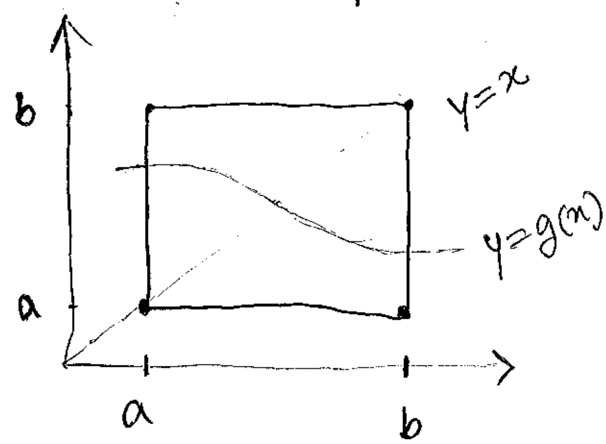
Theorem on the existence of a fixed point

if $g \in C[a,b]$ (refers to continuity) and $g(x) \in [a,b]$ for all $x \in [a,b]$

then, g has at least one fixed point in $[a,b]$.

In addition, if, $g'(x)$ exists on (a,b) and a positive constant "K" <1 exists with $|g'(x)| \leq K$, for all $x \in (a,b)$

then there's exactly one fixed point in $[a,b]$



$|g'(x)| \leq K$ is the condition, where $K<1$. So

$|g'(x)| < 1$

For Part b,

Let's revisit the theory.

if $|g'(x)| \leqslant k$, for all $x \in a, b$ is satisfied,

we can have a corollary stating about the error bound.

Corrolary: Given that above theorem satisfies, error bounds for the error involved in using P_n to approximate p are given by

$$|P_n - p| \leqslant k^n \max\{P_0 - a, b - P_0\}$$

and

$$|P_n - p| \leqslant \frac{k^n}{1-k} |P_1 - P_0|, \text{ for all } n \geqslant 1$$

Example: Given $f(x) = \frac{x^2+3}{5}$, interval $\in [0, 1]$

$$f(0) = \frac{3}{5}, \quad f(1) = \frac{1+3}{5} = \frac{4}{5} < 1$$

We have
$$f'(x) = \frac{d}{dx} \frac{x^2+3}{5} = \frac{2x}{5}$$

$$f'(0) = \frac{2 \cdot 0}{5} = 0 \quad \Big| \quad f'(1) = \frac{2}{5}$$

So,
$$K = \max_{x \in 0,1} \{|f'(x)|\} = \frac{3}{5}$$

According to thr^m

~~being~~ approximated value

$$|x - x^*| \leqslant \frac{K}{1-K} |x_1 - x_0|$$

So, $$\frac{K^n}{1-K}|x_1-x_0| = \frac{\left(\frac{2}{5}\right)^n}{1-\frac{2}{5}} \cdot 1 = \frac{5}{3}\left(\frac{2}{5}\right)^n$$

Now, if error needs to be smaller than

$$\frac{5}{3}\cdot\left(\frac{2}{5}\right)^n \leq 10^{-4}$$

$$\Rightarrow \quad n \approx 8.1$$

suggesting about 9 iteration

Coming back to the HW Problem 4, Part b.

Given $f(x) = 2^{-x}$ interval $\in [1/3, 1]$

$$\Rightarrow \quad f(1/3) = 2^{-1/3} = \frac{1}{2^{1/3}} = 0.7937$$

$$f(1) = 2^{-1} = 0.5$$ decreasing fn^c

Here,

$$f'(x) = -2^{-x}\ln(2)$$

So, $$f'(1/3) = -2^{-1/3}\ln(2) = -0.7937 \times \ln 2$$

$$= -0.5501$$

$$f'(1) = -2^{-1}\ln(2) = -0.3465$$

$$K = \max_{x\in 1/3,1} |f'(x)| = \frac{1}{3}$$

So, $$\frac{K^n}{1-K}|x_1-x_0| = \frac{\left(\frac{1}{3}\right)^n}{(1-K)}\cdot\frac{2}{3} = \frac{\left(\frac{1}{3}\right)^n}{\frac{2}{3}}\cdot\frac{2}{3}$$

$$\Rightarrow \quad \left(\frac{1}{3}\right)^n \leq 10^{-4}$$

That's error 10^{-4} must be less than –

$$\left(\frac{1}{3}\right)^n \leq 10^{-4}$$

$$\Rightarrow \quad n \ln\left(\frac{1}{3}\right) \leq \ln 10^{-4}$$

$$\Rightarrow \quad n \ln\left(\frac{1}{3}\right) \leq \ln\left(10^{-4}\right)$$

$$\Rightarrow \quad n \leq \frac{\ln\left(10^{-4}\right)}{\ln\left(\frac{1}{3}\right)} \approx 8.38$$

So, around 9 steps needed