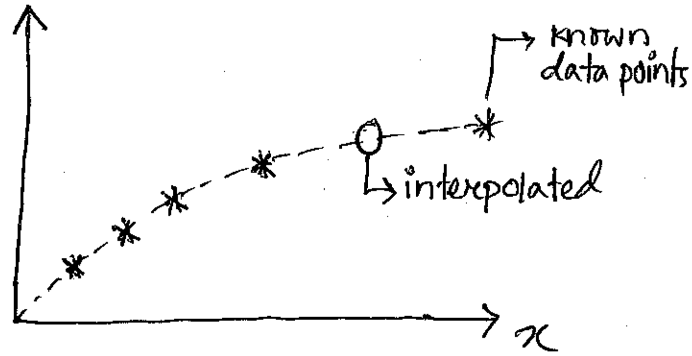
INTERPOLATION

It is the method to construct new data points for a given set of known discrete data points.

Consider the below example

Here, "$*$" are the known discrete data points

"O" is the interpolated valve.



That's interpolation provides approximated valves of a fn^c at any intermediate points.

Types of Interpolation:

- Piecewise constant — assign same value for nearest point.
- Linear interpolation
- Polynomial Interpolation — We cover this in this course.
- Spline interpolation

Polynomial:

- It's a mathematical expression that involves sum of powers in one or more variables multiplied by coefficients.

Assuming constant coefficients, polynomial of one variable —

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots \dots + a_n x^n$$

two variable

$$a_{00} + a_{01} y + a_{10} x + a_{11} xy + a_{12} xy^2 + a_{21} x^2 y + a_{22} x^2 y^2$$

LAGRANGE FORM

Let's assume

$x_0, x_1, x_2, \ldots \ldots x_n$ are $(n+1)$ interpolating points

nodes

↳ May, or may not be, uniformly spaced.

$f_0, f_1, f_2 \ldots \ldots f_n$ are fn^c values

Generally, a polynomial of degree at most "n", P_n, satisfies $P_n(x_i) = f_i$ for $i = 0, 1, 2 \ldots \ldots n$

Linear interpolation:

Let's consider only two points x_0, x_1. corresponding fn^c values are f_0 and f_1.

So

$$P_1(x) = a_0 + a_1 x$$

↳ degree of polynomial

For, $x = x_0$

$$P_1(x_0) = a_0 + a_1 \cdot x_0 = f_0 \cdots (1)$$

$$P_1(x_1) = a_0 + a_1 x_1 = f_1 \cdots (2)$$

Now, (2) − (1)

$$f_1 - f_0 = \cancel{a_0} + a_1 x_1 - \cancel{a_0} - a_1 x_0 = a_1(x_1 - x_0)$$

$$\Rightarrow a_1 = \frac{f_1 - f_0}{x_1 - x_0}$$

$$a_0 = f_0 - a_1 \cdot x_0$$

$$= f_0 - \left(\frac{f_1 - f_0}{x_1 - x_0}\right) x_0$$

$$= f_0 - \left(\frac{f_1 x_0 - f_0 x_0}{x_1 - x_0}\right)$$

$$= \frac{f_0 x_1 - \cancel{f_0 x_0} - f_1 x_0 + \cancel{f_0 x_0}}{x_1 - x_0}$$

$$= \frac{f_0 x_1 - f_1 x_0}{x_1 - x_0}$$

Therefore,

$$P_1(x) = \frac{f_0 x_1 - f_1 x_0}{x_1 - x_0} + \frac{f_1 - f_0}{x_1 - x_0} x$$

$$= \frac{f_0 x_1}{x_1 - x_0} - \frac{f_0 x}{x_1 - x_0} + \frac{f_1 x}{x_1 - x_0} - \frac{f_1 x_0}{x_1 - x_0}$$

$$= \frac{x - x_1}{x_0 - x_1} f_0 + \frac{x - x_0}{x_1 - x_0} f_1$$

Now, with $L_{1,0}(x)$ and $L_{1,1}(x)$, we can rewrite -

$$P_1(x) = L_{1,0}(x)\, f_0 + L_{1,1}(x)\, f_1 = \sum_{i=0}^{1} L_{1,i}(x)\, f_i$$

Worthwhile to mention :

$$L_{1,0}$$

Associated interpolating Point.

Degree of the polynomial

Definition :

The Lagrange polynomial $L_{n,j}(x)$ has degree "n" and is associated with interpolating point x_j in the sense -

$$L_{n,j}(x_i) = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$$

With this family of functions, we can demonstrate that —

$$P_n(x) = \sum_{i=0}^{n} L_{n,i}(x)\, f_i \quad \cdots \quad \cdots \quad (1)$$

$$= 0 \cdot f_0 + \cdots + f_{j-1} \cdot 0 + f_j \cdot 1 + 0 \cdot f_{j+1} + \cdots$$

$$= f_j$$

Since, (1) is based on Lagrange polynomials, it is referred to as the Lagrange form of the interpolating polynomials.

⊞ However,

we need explicit formulas for the $L_{n,j}$

$L_{n,j} \rightarrow$ n^{th} degree polynomial

As $L_{n,j}$ is an n^{th} degree polynomial with n-roots located at $x = x_i \ (i \neq j)$, $L_{n,j}$ must be of the form

$$c(x-x_0)(x-x_1)\cdots(x-x_{j-1})(x-x_{j+1})\cdots(x-x_n)$$

Here, c is some constant. $c = ?$

Using the fact that $L_{n,j}(x_j) = 1$, we can get

$$c = \frac{1}{(x_j-x_0)(x_j-x_1)\cdots(x_j-x_{j-1})(x_j-x_{j+1})\cdots(x_j-x_n)}$$

Finally,

$$L_{n,j}(x) = \frac{(x-x_0)(x-x_1)\cdots\cdots(x-x_{j-1})(x-x_{j+1})\cdots(x-x_n)}{(x_j-x_0)(x_j-x_1)\cdots(x_j-x_{j-1})(x-x_{j+1})\cdots(x_j-x_n)}$$

$$= \prod_{i=0,\, i\neq j}^{n} \frac{x-x_i}{x_j-x_i}$$

$\boxplus$ But how do we calculate the coefficients ?

We can ~~create~~ use interpolating conditions to create a system of linear condition.

However, this is time consuming and cumbersome process to try with.

$\downarrow$ more efficient approach

LAGRANGE POLYNOMIAL

We know from the linear interpolating polynomial—

$$P_1(x) = \frac{x-x_1}{x_0-x_1} f_0 + \frac{x-x_0}{x_1-x_0} f_1$$

Polynomial degree one	Polynomial degree one
@ $x=x_0$ value $=1$	@ $x=x_0$ value $=0$
@ $x=x_1$ value $=0$	@ $x=x_1$ value $=1$

Observations : Coefficients are polynomials of the same degree as the overall interpolating polynomial

also,

$f_0 = 1$ at $x = x_0$, 0 otherwise

$f_1 = 1$ at $x = x_1$ 0 otherwise

These coefficient polynomials are known as Lagrange polynomials. Denoted as

$$L_{1,0}(x) \doteq \frac{x-x_1}{x_0-x_1} \quad \text{and} \quad L_{1,1}(x) = \frac{x-x_0}{x_1-x_0}$$

Example

Determine the linear Lagrange polynomial that passes through points $(2,4)$ and $(5,1)$

$$x_0 = 2, \quad x_1 = 5$$
$$f(x_0) = 4, \quad f(x_1) = 1$$

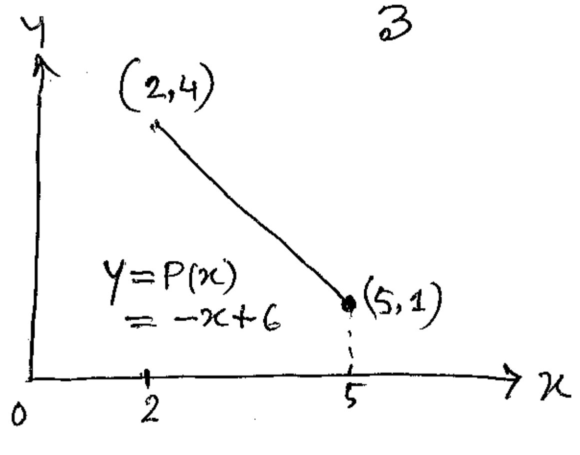
So,

$$L_0(x) = \frac{x-5}{2-5} = -\frac{1}{3}(x-5)$$

$$L_1(x) = \frac{x-2}{5-2} = \frac{1}{3}(x-2)$$

So,

$$P(x) = -\frac{1}{3}(x-5)\cdot 4 + \frac{1}{3}(x-2)\cdot 1$$
$$= -\frac{4}{3}(x-5)\cdot 1 + \frac{1}{3}(x-2)$$
$$= \frac{-4x+20+x-2}{3} = \frac{-3x+18}{3}$$
$$= -x+6$$



Disadvantage of Linear interpolation:

- It is very quick and easy.

  But

  not very precise

- Not differentiable at point x_K

Linear interpolation is computed using only two data points.

However,

If more than two data points are available

↓

We can have a higher-degree interpolating polynomial

- Suppose, n+1 data points are available

  ↓ each point translates to

  one interpolation condition. So, we have n+1 interpolation condition.

  ↓

  Allows calculation of n+1 polynomial coefficient.

- Recall: n^{th}-degree polynomial has n+1 coefficients

  ↓ suggests

  (n+1) coefficients/data points can determine a polynomial of degree at most "n".

NEWTON FORM OF INTERPOLATION

A number of drawbacks may arise

- Amount of computation needed for higher order polynomial is large.
- Over-fitting may be a problem; this happens due to their rigidity. Rigidity relates to smoothness.

Runge's phenomenon

measured by the total number of derivatives that are continuous.

Here in Newton's approach

N^{th} degree interpolating polynomial is obtained by $(N+1)$ data points.

Types of Newton form of Interpolation

1. Forward difference
2. Backward difference

Forward difference :

First order difference

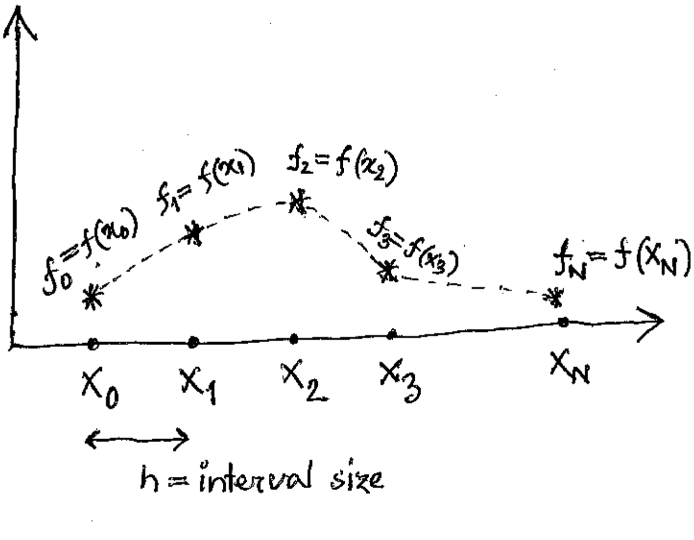
$$\Delta f_i = f_{i+1} - f_i$$

Second order difference

$$\Delta^2 f_i = \Delta f_{i+1} - \Delta f_i$$

$$= (f_{i+2} - f_{i+1}) - (f_{i+1} - f_i)$$

$$= f_{i+2} + f_i - 2f_{i+1}$$



So, to approximate $f(x)$, we must evaluate f_0^1; f_0^2, and so on.

We know $\Delta f_0 = f_1 - f_0$ | where, $f_1 = f(x_1)$

Using Taylor series to approximate

$$f_1 = f_0 + (x_1 - x_0) f_0^{(1)} + \frac{(x_1 - x_0)^2 \cdot f_0^{(2)}}{2!} + \dots \dots$$
$$= f_0 + h f_0^{(1)} + \frac{h^2}{2!} \cdot f_0^{(2)} + \dots \dots O(h)^3$$
$$= \underline{\qquad\qquad} + \frac{1}{3!}(x_3 - x_0)^3 f_0^{(3)} + \dots$$

So,

$$\Delta f_0 = f_0 + h f_0^{(1)} + \frac{h^2}{2!} f_0^{(2)} - f_0$$
$$= h f_0^{(1)} + \frac{h^2}{2!} f_0^{(2)} + O(h)^3$$
$$= h f_0^{(1)} + \frac{h^2}{2!} f_0^{(2)} + \frac{1}{3!} h^3 f_0^{(3)} + O(h)^4$$
$$\Rightarrow f_0^{(1)} = \frac{\Delta f_0}{h} - \frac{h^2}{2} f_0^{(2)} - O(h^{\cancel{3}})^{2}$$
$$= \frac{\Delta f_0}{h} - \frac{1}{2!} h f_0^{(2)} - \frac{1}{3!} h^2 f_0^{(3)} - O(h^3)$$

Second order forward difference

$$\Delta^2 f_0 = f_2 - 2f_1 + f_0$$

| We have to write f_1 and f_2 in terms of f_0.

$f_1 = f(x_1)$ is approximated previously.

So, For $f_2 = f(x_2)$ we use Taylor series to express f_2 interms of f_0 and derivatives evaluated at x_0.

$$f_2 = f_0 + (x_2 - x_0) f_0^{(1)} + \frac{1}{2!}(x_2 - x_0)^2 f_0^{(2)} + \dots \dots \dots \quad (1)$$

Here, $x_2 - x_0 = 2h$

$$f_2 = f_0 + 2h \cdot f_0^{(1)} + \frac{1}{2!}(2h)^2 f_0^{(2)} + \dots \dots + O(h)^3$$
$$= f_0 + 2h \cdot f_0^{(1)} + h^2 \frac{4}{2!} f_0^{(2)} + \dots \dots + O(h)^3 \quad \left(+ \frac{8}{3} h^3 f_0^{(3)}\right) + O(h)^4$$

So,

$$\Delta^2 f_0 = f_2 - 2f_1 + f_0$$

$$\Rightarrow \Delta^2 f_0 = f_0 + 2h \cdot f_0^{(1)} + \frac{4}{2!} f_0^{(2)} + \dots - 2 \cdot \left[f_0 + h f_0^{(1)} + \frac{h^2}{2!} f_0^{(2)} + \dots \right] + f_0$$

$$= \cancel{f_0} + \cancel{2h \cdot f_0^{(1)}} + \frac{4}{2!} f_0^{(2)} h^2 - \cancel{2 f_0} - \cancel{2h f_0^{(1)}} - h^2 f_0^{(2)} - \dots + \cancel{f_0}$$

$$= 2 \cdot h^2 \cdot f_0^{(2)} - h^2 f_0^{(2)} + O(h)^3$$

$$\Rightarrow f_0^{(2)} = \frac{\Delta^2 f_0}{h^2} + O(h)$$

$$= \frac{\Delta^2 f_0}{h^2} - h f_0^{(3)} + O(h)^2$$

depends on where we stop in f_1 approximation

This comes when f_2 is approximated considering upto $\frac{(x_2 - x_0)^3}{3!} f_0^{(3)} + \dots$ in Eq. 1

Similarly,

$$f_0^{(3)} = \frac{\Delta^3 f_0}{h^3} + O(h)$$

$$\vdots = \frac{\Delta^3 f_0}{h^3} + O(h)$$

Consider all these values and plug in to $f(x)$ approximation :

$$f(x) = f(x_0) + (x - x_0) \frac{df}{dx}\Big|_{x=x_0} + \frac{(x-x_0)^2}{2!} \frac{d^2 f}{dx^2}\Big|_{x=x_0} \dots \dots$$

$$= f_0 + (x - x_0) f_0^{(1)} \cancel{(x_0)} + \frac{(x-x_0)^2}{2!} f_0^{(2)} + \dots$$

$$= f_0$$

Third order difference

$$
\begin{aligned}
\Delta^3 f_i &= \Delta^2 f_{i+1} - \Delta^2 f_i \\
&= f_{i+1+2} + f_{i+1} - 2f_{i+1+1} - \left(f_{i+2} + f_i - 2f_{i+1}\right) \\
&= f_{i+3} + f_{i+1} - 2f_{i+2} - f_{i+2} - f_i + 2f_{i+1} \\
&= f_{i+3} + 3f_{i+1} - 3f_{i+2} - f_i
\end{aligned}
$$

Now, the difference table looks as below:

$i \rightarrow$ data point	f_i	Δf_i	$\Delta^2 f_i$ $\rightarrow$ order	$\Delta^3 f_i$ order
0	f_0	$\Delta f_0 = f_1 - f_0$	$\Delta^2 f_0 = \Delta f_1 - \Delta f_0$	$\Delta^3 f_0 = \Delta^2 f_1 - \Delta^2 f_0$
1	f_1	$\Delta f_1 = f_2 - f_1$	$\Delta^2 f_0 = \Delta f_2 - \Delta f_1$	$\Delta^3 f_1 = \Delta^2 f_2 - \Delta^2 f_1$
2	f_2	$\Delta f_2 = f_3 - f_2$	$\Delta^2 f_2 = \Delta f_3 - \Delta f_2$	
3	f_3	$\Delta f_3 = f_4 - f_3$		
4	f_4			

As we see, order of the difference that can be calculated depends directly on the number of points available.

Derivation of Newton Forward Interpolation

(assuming equi-spaced points, not mandatory though)

Let's consider the Taylor's series expansion of $f(x)$ about x_0

$$
f(x) = f(x_0) + (x-x_0)\left.\frac{df}{dx}\right|_{x=x_0} + \frac{(x-x_0)^2}{2!}\frac{d^2f}{dx^2} + \frac{(x-x_0)^3}{3!}\frac{d^3f}{dx^2} + \cdots \cdots
$$

$$
= f(x_0) + \underbrace{(x-x_0)}_{h} f^1(x_0) + \frac{(x-x_0)^2}{2!} f^2(x_0) + \cdots\cdots
$$

We get,

$$f(x) = f_0 + (x-x_0)\frac{\Delta f_0}{h} + \frac{1}{2!}(x-x_0)\left[-h+(x-x_0)\right]\frac{\Delta^2 f_0}{h^2}$$
$$+ \frac{1}{3!}(x-x_0)\left[2h^2+(x-x_0)^2 - 3(x-x_0)h\right]\frac{\Delta^3 f}{h^3}$$
$$+ O(h)^4 + H.O.T$$

$$\Rightarrow \quad f(x) = f_0 + (x-x_0)\frac{\Delta f_0}{h} + \frac{1}{2}(x-x_0)\left[x-(x_0+h)\right]\frac{\Delta^2 f_0}{h^2}$$
$$+ \frac{1}{3!}(x-x_0)\left[\left(x-(x_0+h)\right)(x_0+2h)\right]\frac{\Delta^3 f_0}{h^3}$$
$$+ O(h)^4 + H.O.T$$

We know. $\quad x_0 + h = x_1, \quad x_0 + 2h = x_2$