INITIAL VALUE PROBLEM

Many real life problems can be manifested using mathematical equations.

An equation in which the unknown function appears inside one or more derivatives is called a differential equation.

When the unknown fn^c is dependent on only one independent variable, the eq^n is differential equation (ordinary)

When a set of initial condition given, we call it initial value problem

When the unknown fn^c depends on more than one independent variable, we call it partial differential equation.

if additional conditions are specified at more than one value of independent variable we call it Boundary Value Problem

INITIAL VALUE PROBLEM

General form of Eq^n

$$y'(t) = \frac{d}{dt} y(t) = f(t, y(t))$$

$$y(a) = \alpha, \qquad a \leqslant t \leqslant b$$

initial value

Here, right hand side fn^c "f" and the initial value "α" are given.

Example system:

Let us consider $P(t)$, denoting the population of the species at any time t.

Suppose, initial population $P(0) = P_0$

A standard assumption in population model is that the instantaneous rate of change in the population is proportional to the population —

$$\frac{dP}{dt} = KP$$

We assume, K could be constant

$$\Rightarrow \frac{dP}{P} = \frac{1}{K} dt$$

$$\Rightarrow \int \frac{dP}{P} = \frac{1}{K} \int dt$$

$$\Rightarrow \ln P = \frac{t}{K} + c$$

integral constant

at $t = 0$

$$\ln P_0 = \frac{0}{K} + c = c$$

$$\Rightarrow \ln P_0 = c$$

$$\ln P = \frac{t}{K} + \ln P_0$$

$$\Rightarrow \ln P - \ln P_0 = \frac{t}{K}$$

$$\Rightarrow \ln P/P_0 = \frac{t}{K}$$

$$\Rightarrow P/P_0 = \ln^{-1}\left(\frac{t}{K}\right) = e^{-t/K}$$

$$\Rightarrow P = P_0 e^{-t/K}$$

However, if K is in fn^c form and has direct dependency on P, we obtain form of eq^{n}s that are often analytically in-tractable

Precisely,

$$\frac{dP}{dt} = \left[r - \lambda P - c\int_0^t P(\tau)\,d\tau\right] P$$

This can also be transformed to an initial value problem.

An initial value problem that has

- a unique solution, and
- is stable

is said to be well-posed problem.

one way to decide if it is well-posed is the

Lipschitz condition

Lipschitz condition

A fn^c $f(t,y)$ satisfies a Lipschitz condition in y on set $D \subset \mathbb{R}^2$ if there exists a constant $L > 0$ such that

$$|f(t,y_1) - f(t,y_2)| \leq L|y_1 - y_2|$$

Example : Given. $f(t,y) = y \sin t$, $D = \mathbb{R}^2$

So, $f(t,y_1) - f(t,y_2) = y_1 \sin t - y_2 \sin t$

$$= (y_1 - y_2)\sin t$$

So, $|f(t,y_1) - f(t,y_2)| = |\sin t|\,|(y_1 - y_2)|$

$$\leq |y_1 - y_2|$$

Thus $|f(t,y_1) - f(t,y_2)| \leq 1 \cdot |y_1 - y_2|$ Lipschitz constant

for all $(t,y_1), (t,y_2) \in D$

Hence, f satisfies a Lipschitz condition.

However, the definition only may be difficult to show that a fn^c satisfies a Lipschitz condition

The below results / Theorem may be useful in certain instances :

Theorem :

If "f" is defined on $D = \{(t,y) \mid a \leq t \leq b, c \leq y \leq d\}$ and there exists a constant $L > 0$ such that

$$\left| \frac{\partial f}{\partial y}(t,y) \right| \leq L$$

for all $(t,y) \in D$, then f satisfies a Lipschitz condition in y on the set D with Lipschitz constant L

Example: Given $\frac{dy}{dt} = y - t^2 + 1$, $0 \leq t \leq 2$, $y(0) = 0.5$

$D = \{(t,y) \mid 0 \leq t \leq 2$ and $-\infty < y < \infty\}$

Here, $f(t,y) = y - t^2 + 1$

$\Rightarrow \frac{d}{dt} f(t,y) = -2t$ | $\frac{d f(t,y)}{dy} = 1$

So, $\left| \frac{d f(t,y)}{dy} \right| = |1| = 1$

So, $f(t,y)$ satisfies a Lipschitz condition in y on D.

Well-posed

Suppose $D = \{(t, y) \mid a \leq t \leq b \text{ and } -\infty < y < \infty\}$

If f is continuous and satisfies a Lipschitz condition in the variable y on the set D, then the initial-value problem

$$\frac{dy}{dt} = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha$$

is well-posed.

Euler's Method

- Most elementary technique to approximate the solution of initial value problem
- Simplicity of this technique is often helpful to understand the complex problem.

Consider a scalar, first order, initial value problem:

$$y'(t) = \frac{d}{dt}y(t) = f(t, y(t)), \qquad a \leq t \leq b$$

$$y(a) = \alpha$$

Solution : $y(t)$

Our goal : approximate $y(t)$.

We calculate w at discrete time point

$$a = t_0 < t_1 < t_2 < t_3 \quad \cdots \quad < t_N = b$$

So, Notationally

$$w_i \approx y_i = y(t_i)$$

Approximated y at time t_i

For simplicity,

Approximate solution will be sought at equally spaced points.

total points considered

N.

Thus, we can define step size:

$$h = (b-a)/N$$

and the t_i can be written as

$$t_i = a + ih \quad (i = 0, 1, 2 \ldots \ldots N)$$

Derivation of Euler Method:

Let's assume that the true solution $y(t)$ has two continuous derivatives on $[a, b]$

Now, expanding the true solution $y(t)$ about the point $t = t_i$ produces

$$y(t) = y_i + (t - t_i)\, y_i' + \frac{1}{2!}(t - t_i)^2\, y''(\xi)$$

where, ξ is guaranteed to lie between t and t_i

Now, at $t = t_{i+1}$

$$y(t_{i+1}) = y_i + (t_{i+1} - t_i)\, y_i' + \frac{1}{2!}(t_{i+1} - t_i)^2\, y''(\xi)$$

$$\Rightarrow \quad y(t_{i+1}) = y_i + h y_i' + \frac{1}{2} h^2\, y''(\xi)$$

By neglecting the error term,

$$y_{i+1} = y_i + h f(t_i, y_i)$$

Cosidering the approximate solution,

$$y_{i+1} = w_{i+1}, \quad y_i = w_i$$

we obtain,

$$w_{i+1} = w_i + h f(t_i, w_i) \quad \text{for each } i = 0, 1, 2, \ldots N-1$$

Example :

Given $y' = y - t^2 + 1$, $0 \leq t \leq 2$.

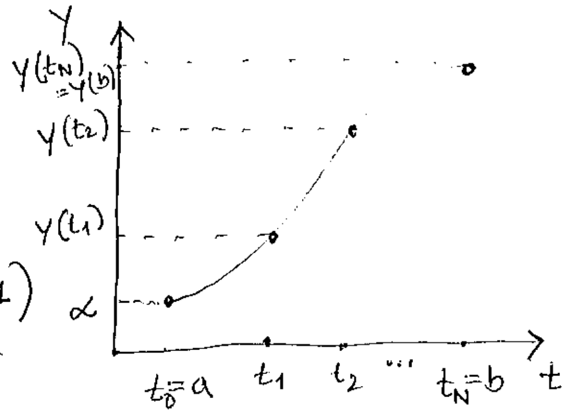
$y(0) = 0.5$

Use approximate sol^n as calculated in Euler Method

Sol^n : consider $h = 0.5$. $f(t,y) = y - t^2 + 1$

$w_0 = y(0) = 0.5$ $\Rightarrow f(t_0, y_0) = w_0 - (0.0)^2 + 1$
$= 0.5 + 1$

$$w_1 = w_0 + h f(t_0, y_0)\Big|_{i=0}$$
$$= 0.5 + 0.5(1.5)$$
$$= 1.25$$



$$w_2 = w_1 + 0.5(1.25 - (0.5)^2 + 1)$$
$$= 1.25 + 0.5(2.0) = 2.25$$

$$w_3 = 2.25 + 0.5(2.25 - 0.5(1)^2 + 1)$$
$$= 2.25 + 0.5(2.25) = 3.375$$

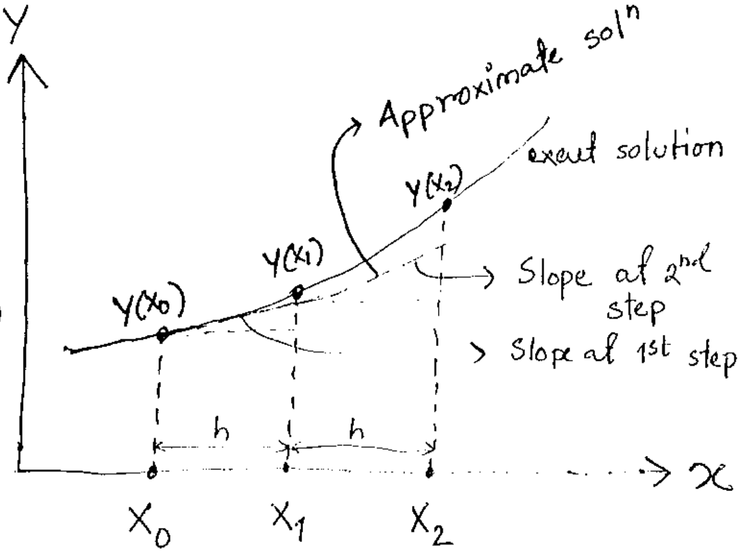
and

$$w_4 = w_3 + 0.5(w_3 - (1.5)^2 + 1) = 3.375 + 0.5(2.125)$$
$$= 4.4375$$

Euler's Method : Geometric interpretation

Consider the below graph

In Euler method, curve between x_0 and x_1 is approximated by the straight line passing through (x_0, y_0). Slope: $f(x_0, y_0)$



The curve between x_1 and x_2 is approximated by the straight-line passing through (x_1, y_1). The slope of the line is $f(x_1, y_1)$

Error at step n is $|Y(x_n) - x_n|$

☐ Given the IVP

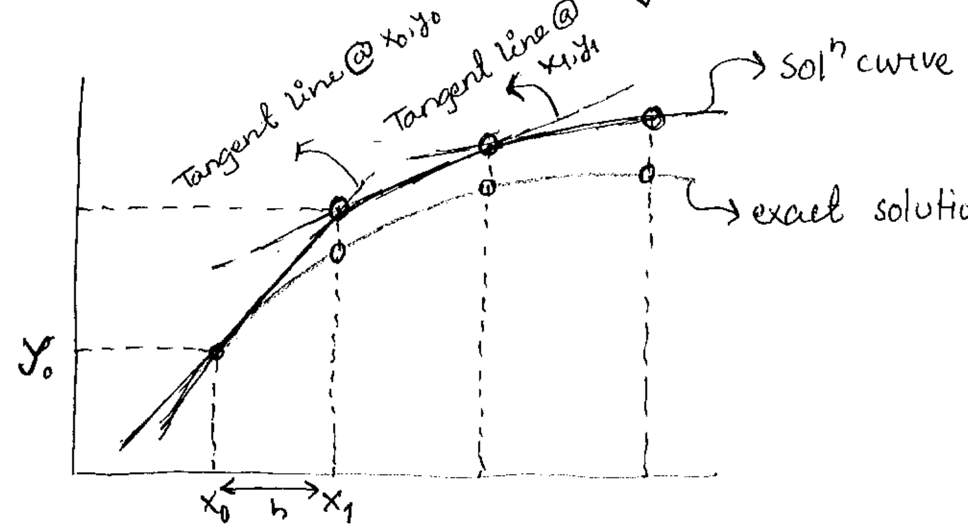
$$\frac{dy}{dx} = f(x, y)$$
$$Y(x_0) = y_0$$

here, it could be $\frac{dy}{dt}$ as well

Let's assume that x_0 is the initial point. We approximate the solution to the IVP, at $x = x_1 = x_0 + h$

idea behind Euler's Method

is to use tangent line



Tangent line is drawn at (x_0, y_0), and the eq^n becomes

$$y(x) = y_0 + m(x - x_0)$$

Slope at (x_0, y_0)

From the given problem, we can write

$$\frac{dy}{dx} = f(x_0, y_0)$$

This is "m"

$$Y(x) = Y_0 + f(x_0, Y_0)(x - x_0)$$

since, $x = x_1$

$$Y(x) = Y_0 + f(x_0, Y_0)(x_1 - x_0)$$

$$\Rightarrow \quad Y_1 = Y_0 + h\, f(x_0, y_0)$$

Now, we wish to obtain exact solution at

$$x_2 = x_1 + h$$

We draw a tangent line to the solution curve through (x_1, y_1).

So, the slope of this tangent line is $f(x_1, y_1)$. So, the eqⁿ becomes,

$$y(x) = y_1 + f(x_1, y_1)(x - x_1)$$

$$\Rightarrow y_2 = y_1 + f(x_1, y_1)(x_2 - x_1)$$

$$\Rightarrow y_2 = y_1 + h\, f(x_1, y_1)$$

In general, $y_{n+1} = y_n + h\, f(x_n, y_n)$

Modified Euler's Method:

Accuracy is an important issue in any approximation algorithm.

$\downarrow$ Predictor-corrector Method

Modified Euler's Method.

→ We improve the approximation by once more applying Euler's Method.

In Modified Euler's Method:

It uses Euler's Method formula to obtain First Approximation

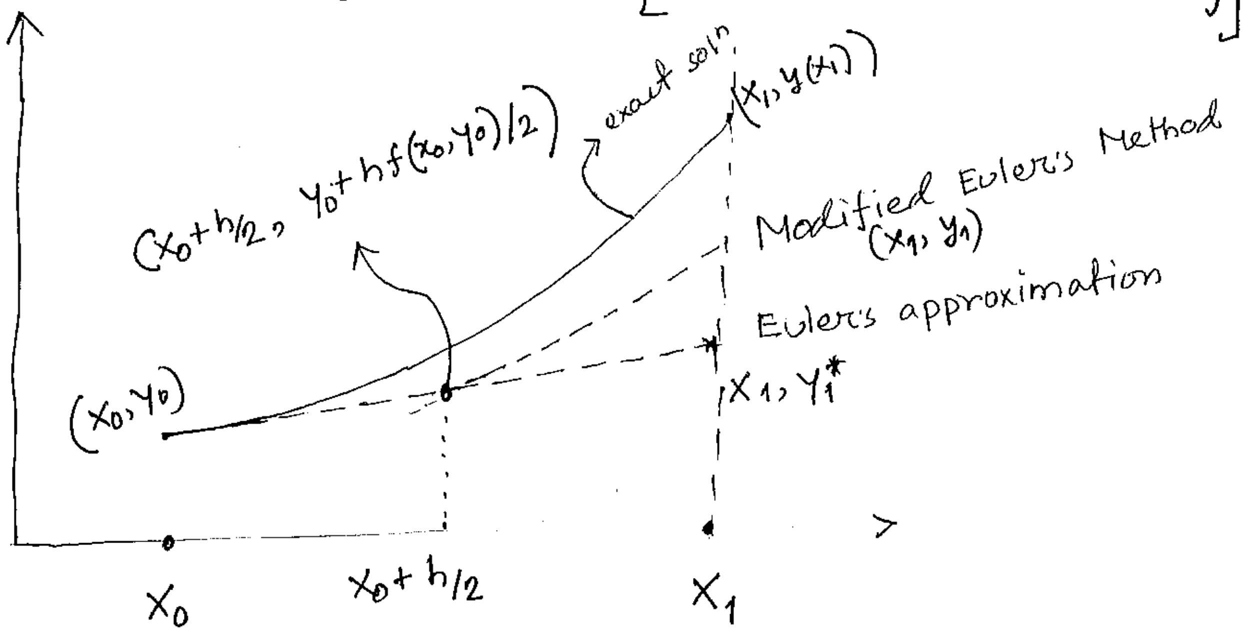
$$y^*_{n+1} = y_n + f(x_n, y_n) \cdot h$$

This first approximation is corrected/improved by applying Euler's Method once more.

This step

Uses average of the slopes of the solution curves through (x_n, y_n) and (x_{n+1}, y^*_{n+1}). This gives—

$$y_{n+1} = y_n + \frac{1}{2}h\left[f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})\right]$$



For $n=1$, we interpret using modified Euler approximations arriving from first stepping to the point —

$$P\left(\underset{1}{x_n} + \frac{h}{2},\ y_n + \frac{hf(x_n, y_n)}{2}\right)$$

In summary, Modified Euler Method for approximating the solution to the IVP :

$$y' = f(x, y), \quad y(x_0) = y_0$$

at the points $x_{n+1} = x_0 + nh \quad (n = 0, 1, 2, \dots)$

$$y_{n+1} = y_n + \frac{1}{2}h\left[f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})\right]$$

Where,

$$y^*_{n+1} = y_n + h f(y_n, x_n) \; ; \; n = 0, 1, 2 \dots$$

↳ obtained using Euler Method

One-Step versus Multistep Methods

- Initial Value Problem (IVP) solvers can be classified into two main categories
  - One-step Method
  - Multi-step Method

One-step method:

General form of one-step method is

$$\frac{w_{i+1} - w_i}{h_i} = \phi\left(f, t_i, w_i, w_{i+1}, h_i\right) \quad \cdots\cdots (1)$$

where, $h_i = t_{i+1} - t_i$

As we see, from (1), computation of w_{i+1} (value at the next iteration) requires the knowledge of w_i only. Here, w_i is one-step prior to w_{i+1}.

When ϕ depends on w_{i+1}, the method is implicit method.

* Now, if the fn^c ϕ is independent of w_{i+1}, the method is explicit.

* In explicit method, knowing the R.H.S fn^c "f" and the given values t_i, w_i, h_i is sufficient to calculate w_{i+1}.

* Euler's method is an explicit method.

One-step continues ...

Consider the difference equation of Euler's Method :

$$\frac{w_{i+1} - w_i}{h_i} = f(t_i, w_i)$$

As we see, $\phi(f, t_i, h_i, w_{i+1}, w_i) = f(t_i, w_i)$

no $w_{i+1} \rightarrow$ explicit
only $w_i \rightarrow$ One-step

Multi-step :

One-step methods use information regarding where we are currently now to predict where we should be at the next time step.

Whereas,

Multi-step methods use information regarding where we are and where we have been to make that prediction.

$\downarrow$ so, two-step method use

Both w_i, w_{i-1}

$\downarrow$ 3-step method

w_i, w_{i-1}, w_{i-2}

$\downarrow$ n-step method

The method is explicit when $b_0 = 0$.

$$\frac{w_{i+1} - a_1 w_i - a_2 w_{i-1} - \cdots \cdots \cdots - a_m w_{i+1-m}}{h_i} =$$

$$b_0 f(t_{i+1}, w_{i+1}) + b_1 f(t_i, w_i) + \cdots \cdots + b_m f(t_{i+1-m}, w_{i+1-m})$$

Higher-order One-step Methods:

Let's consider the IVP

$$y'(t) = f(t, y(t)), \quad a \leq t \leq b$$

$$y(a) = \alpha$$

↳ initial value

Assuming $y(t)$ has two continuous derivatives — we can obtain using Taylor's series about $t = t_i$

$$y(t) = y_i + (t - t_i) y_i' + \frac{1}{2}(t - t_i)^2 y''(\xi)$$

$$\Rightarrow \quad y(t) = y_i + h\, f(t_i, y_i) + \frac{1}{2} h^2 y''(\xi)$$

Now, for higher order, we assume that the true solutions has more continuous derivatives.

For instance, if we assume y has $n+1$ continuous derivatives on $[a, b]$, using Taylor series we obtain

$$Y(t) = y_i + (t - t_i) y_i' + \frac{(t - t_i)^2}{2} y_i'' + \cdots \cdots + \frac{(t - t_i)^n}{n!} y_i^{(n)} + \frac{(t - t_i)^{n+1}}{(n+1)!} y^{(n+1)}(\xi_i)$$

Here, we can replace y_i' by the term $f(t_i, y_i)$

So, this suggests that derivative of $f(t_i, y_i)$ will have to be ~~t~~ calculated as well.

↳ disadvantages for higher-order Taylor method.

Second-order Runge-Kutta Method

Let's consider the explicit one-step method

$$\frac{w_{i+1} - w_i}{h} = \phi(f, t_i, w_i, h) \quad \cdots \cdots (1) \qquad h = t_{i+1} - t_i$$

with $$\phi(f, t, w, h) = a_1 f(t, y) + a_2 f(t + \alpha_2, y + \delta_2 f(t, y)) \quad \cdots \cdots (2)$$

Our objective is to determine values for parameter a_1, a_2, α_2 and δ_2, so that $\phi(f, t, h, w)$ provides an $O(h^2)$ approximation tof the R.H.S of the $O(h^2)$ Taylor Method.

In practice, we implement (1) and (2) in two stages.

Stage 1 : $$\tilde{w} = w_i + \delta_2 f(t_i, w_i)$$

This is similar to Euler Method if $\delta_2 = h$. Therefore, $\tilde{w} = y(t_i + \delta_2)$

Stage 2 : We determine w_{i+1} using equation (1) & (2)

So, $$\frac{w_{i+1} - w_i}{h} = a_1 f(t_i, w_i) + a_2 f(t_i + \alpha_2, \tilde{w})$$

Since, $\tilde{w} \approx y(t_i + \delta_2)$, the last term in the eq^n for w_{i+1} suggest that we should select

$$\alpha_2 = \delta_2$$

Second-order Runge-Kutta Methods

A few most common second-order Runge-Kutta methods are as follows:

Modified Euler Method: $a_1 = 0,\ a_2 = 1,\ \alpha_2 = \delta_2 = h/2$

$$\tilde{w} = w_i + \frac{h}{2} f(t_i, w_i)$$
$$w_{i+1} = w_i + h f\left(t_i + \frac{h}{2}, \tilde{w}\right)$$

Heun Method: $a_1 = a_2 = 1/2,\ \alpha_2 = \delta_2 = h$

$$\tilde{w} = w_i + h f(t_i, w_i)$$
$$w_{i+1} = w_i + \frac{h}{2}\left[f(t_i, w_i) + f(t_i + h, \tilde{w})\right]$$

Optimal RK2 Method: $a_1 = 1/4,\ a_2 = 3/4,\ \alpha_2 = \delta_2 = 2h/3$

$$\tilde{w} = w_i + \frac{2h}{3} f(t_i, w_i)$$
$$w_{i+1} = w_i + \frac{h}{4} f(t_i, w_i) + \frac{3h}{4} f\left(t_i + \frac{2h}{3}, \tilde{w}\right)$$

Classical Fourth-Order Runge-Kutta Method

The most common Runge-Kutta method is the classical fourth-order scheme. It updates the approximate solution at each time-step according to the formula

$$w_{i+1} = w_i + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

where,

$$K_1 = hf(t_i, w_i)$$
$$K_2 = hf\left(t_i + \frac{h}{2}, w_i + \frac{K_1}{2}\right)$$
$$K_3 = hf\left(t_i + \frac{h}{2}, w_i + \frac{K_2}{2}\right)$$
$$K_4 = hf(t_i + h, w_i + K_3)$$

- It requires four fn^c evaluation at each time-step
- It provides fourth order accuracy.

Example:

$$\frac{dx}{dt} = 1 + \frac{x}{t} \quad (1 \leq t \leq 6), \; x(1) = 1$$

Use 4^{th} order Runge-Kutta method with a step size $h = 1$.

For the first-time-step, with $t_0 = w_0 = 1$, we calculate K_1, K_2, K_3, K_4.

$$K_1 = hf(t_0, w_0) = hf(1,1) = 2$$
$$K_2 = hf(t_0 + h/2, w_0 + K_1/2) = f(1.5, 2) = 2.3333$$
$$K_3 = hf(t_0 + h/2, w_0 + K_2/2) = f(1.5, 2.1667)$$
$$= 2.4444$$
$$K_4 = hf(t_0 + h, w_0 + K_3)$$
$$= f(2, 3.4444) = 2.72222$$

- Since the linear combination of f values in the w_{i+1} equation has the appearance of a weighted average, we expect to find that —

$$a_1 + a_2 = 1$$

- Example: Second-order Runge-Kutta

$$\frac{dx}{dt} = 1 + \frac{x}{t}, \quad 1 \leq t \leq 6, \quad x(1) = 1.$$

step-size $= h = 0.5$

First time-step: $t_0 = w_0 = 1$. So,

$$f(t_0, w_0) = 1 + \frac{w_0}{t_0} = 1 + \frac{1}{1} = 2$$

So,
$$\left.\begin{aligned} \tilde{w} &= w_0 + \delta_2 f(t_0, w_0) \\ &= 1 + 0.5 \times 2 = 1 + 1 = 2 \end{aligned}\right\} \text{Stage 1}$$

Stage 2
$$f(t_0 + h, \tilde{w}) = 1 + \frac{2 = \tilde{w} \text{ (Stage 1)}}{(t_0 = 1 + h = 0.5)}$$
$$= 1 + \frac{2}{1.5} = 2.3333$$

So,
$$w_1 = w_0 + \frac{h}{2}\left[f(t_0, w_0) + f(t_0 + h, \tilde{w})\right]$$
$$= 1 + \frac{0.5}{2}\left[2 + 2.3333\right]$$
$$= 2.08333$$

We assumed $a_1 = a_2 = \frac{1}{2}$; $\alpha_2 = \delta_2 = h$

For the next time-step

$$t_1 = t_0 + h = 1.5.$$

$$f(t_1, w_1) = 1 + \frac{w_1}{t_1} = 1 + \frac{2.08333}{1.5} = 2.38889$$

and $\tilde{w} = w_i + h f(t_i, w_i)$, here $i = 1$ as we deal t_1

$= 2.08333 + 0.5 \times 2.388889$

$= 3.27778$ Stage 1

In the second stage, we calculate

$$f(t_1 + h, \tilde{w}) = 1 + \frac{x}{t} = 1 + \frac{3.277778}{2}$$

$$= 2.638889$$

So, $w_2 = w_1 + \frac{h}{2}\left[f(t_i, w_i) + f(t_i + h, \tilde{w})\right]$

$$\simeq 2.08333 + \frac{0.5}{2}\left[2.388889 + 2.638889\right]$$

$$= 3.340278$$

For instance,

$$\frac{du_1(t)}{dt} = u_1'(t) = f_1(t, u_1, u_2, u_3 \dots u_m), \quad u_1(a) = \alpha_1$$

$$u_2'(t) = f_2(t, u_1, u_2, u_3 \dots u_m), \quad u_2(a) = \alpha_2$$

$$u_3'(t) = f_3(t, u_1, u_2, u_3 \dots u_m); \quad u_3(a) = \alpha_3$$

$$\vdots$$

$$u_m'(t) = f_m(t, u_1, u_2, u_3 \dots u_m); \quad u_m(a) = \alpha_m$$

Here, * t is the independent variable and we are interested in the range $a \leq t < b$

* $u_1, u_2, u_3 \dots u_m$ are dependent variables

$$\bar{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ \vdots \\ u_m(t) \end{bmatrix}, \quad \bar{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_4 \end{bmatrix}$$

$$f(t, \bar{u}) = \begin{bmatrix} f_1(t, u_1, u_2 \dots u_m) \\ \vdots \\ f_m(t, u_1, u_2 \dots u_m) \end{bmatrix}$$

Then.

$$\bar{u}'(t) = \begin{bmatrix} u_1'(t) \\ \vdots \\ u_m'(t) \end{bmatrix}$$

So, original systems of equation can be written as:

$$\bar{u}'(t) = f(t, \bar{u}), \quad a \leq t \leq b$$

$$\bar{u}(a) = \bar{\alpha}$$

In case of the scalar case, it was

$$u'(t) = f(t, u) \quad \Big| \quad a \leq t \leq b$$

$$u(a) = \alpha$$

Euler Method was

$$w_0 = \alpha, \qquad \frac{w_{i+1} - w_i}{h} = f(t_i, w_i)$$

time-stepping

But for vector case:

$$\bar{w}^{(0)} = \alpha$$

$$\frac{\bar{w}^{(j+1)} - \bar{w}^{(j)}}{h} = f(t_j, \bar{w}^{(j)})$$

time-stepping

Subscript can be used to denote the component within the vector.