System of Linear Equations

Given,

$$x_1 + x_2 \quad + 3x_4 = 4 \quad : E1$$
$$2x_1 + x_2 - x_3 + x_4 = 1 \quad : E2$$
$$3x_1 - x_2 - x_3 + 2x_4 = -3 \quad : E3$$
$$-x_1 + 2x_2 + 3x_3 - x_4 = 4 \quad : E4$$

$$(E2 - 2E1) \rightarrow (E2)$$

produces

$$(2x_1 + x_2 - x_3 + x_4) - 2(x_1 + x_2 + 3x_4) = 1 - 2 \times 4 = -7$$

System becomes

$$x_1 + x_2 \quad + 3x_4 = 4 \quad : E1$$
$$-x_2 - x_3 - 5x_4 = -7 \quad : E2$$
$$-4x_2 - x_3 - 5x_4 = -15 \quad : E3$$
$$3x_2 + 3x_3 + 2x_4 = 8 \quad : E4$$

by performing

$$(E_3 - 4E_2) \rightarrow E3$$
$$(E_4 + 3E_2) \rightarrow E4$$

$$x_1 + x_2 \quad + 3x_4 = 4$$
$$-x_2 - x_3 - 5x_4 = -7$$
$$3x_3 + 13x_4 = 13$$
$$-13x_4 = -13$$

- The system of equations is now in triangular form and can be ~~used~~ solved using the backward-substitution process.

Here, E_4 gives $x_4 = 1$.

We use $x_4 = 1$ to obtain x_3 from E3

$$x_3 = \frac{1}{3}(13 - 13x_4) = \frac{1}{3}(13-13) = 0$$

Now, E2 gives —

$$x_2 = -(-7 + 5x_4 + x_3)$$
$$= -(-7 + 5 + 0) = 2$$

and E1 gives —

$$x_1 = 4 - 3x_4 - x_2 = 4-3-2 = -1$$

So, Sol^n becomes

$$x_1 = -1 \quad x_3 = 0$$
$$x_2 = 2 \quad x_4 = 1$$

- Matrix can be used to represent linear system:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \dots a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \dots a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 \dots a_{nn}x_n = b_n$$

Finally, we find that

$$w_1 = 1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1 + \frac{1}{6}\big(2 + 2(2.3333) + 2(2.44444) + 2.72222\big)$$

$$= 3.379630$$

System of Equations [IVP in vector form]

Let us consider a case

$$\left.\begin{aligned} \frac{dx_1}{dt} &= a_1 x_1 - b_1 x_1^2 + c_1 x_1 x_2 \\ \frac{dx_2}{dt} &= a_2 x_2 - b_2 x_2^2 + c_2 x_1 x_2 \end{aligned}\right\} \text{Predator-Prey Model.}$$

Here, x_1 and x_2 are the interacting population and we need to solve this using IVP method.

- Analysis of such systems require more advanced techniques than the scalar case, as they often tend to exhibit a wider variety of system behavior.

- However, fortunately, numerical analysis requires little more than a notational change from the scalar case.

So, we can represent using

$$\bar{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \text{ and } \bar{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

We also can combine them as follows:

$$[\vec{A}, \vec{b}] \equiv \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ \vdots & & & & b_2 \\ & & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n,n} & b_n \end{array}\right]$$

→ Augmented matrix

↓ We perform elementary row operations to obtain a form as below

$$\left[\begin{array}{ccccc|c} a_{11} & a_{12} & \cdots & \cdots & a_{1n} & a_{1,n+1} \\ 0 & a_{22} & \cdots & \cdots & a_{2n} & a_{2,n+1} \\ \vdots & & \ddots & & & \vdots \\ 0 & \cdots & \cdots & 0 & a_{nn} & a_{n,n+1} \end{array}\right]$$

→ Pivoting

→ Called Gaussian elimination

Now, we can solve the linear system to find the unknowns. The method is called Gaussian elimination with backward substitution

Example

$$
\begin{aligned}
x_1 + x_2 + 3x_4 &= 4 \\
2x_1 + x_2 - x_3 + x_4 &= 1 \\
3x_1 - x_2 - x_3 + 2x_4 &= -3 \\
-x_1 + 2x_2 + 3x_3 - x_4 &= 4
\end{aligned}
$$

Augmented Matrix —

$$
\left[\begin{array}{cccc|c}
1 & 1 & 0 & 3 & 4 \\
2 & 1 & -1 & 1 & 1 \\
3 & -1 & -1 & 2 & -3 \\
-1 & 2 & 3 & -1 & 4
\end{array}\right]
$$

$$\downarrow$$

$$
\left[\begin{array}{cccc|c}
1 & 1 & 0 & 3 & 4 \\
0 & -1 & -1 & -5 & -7 \\
0 & -4 & -1 & -7 & -15 \\
0 & 3 & 3 & 2 & 8
\end{array}\right]
\longrightarrow
\left[\begin{array}{cccc|c}
1 & 1 & 0 & 3 & 4 \\
0 & -1 & -1 & -5 & -7 \\
0 & 0 & 3 & 13 & 13 \\
0 & 0 & 0 & -13 & -13
\end{array}\right]
$$

Now, we use back-substitution and the whole process is known as —

" Gaussian Elimination with backward substitution "

TRANSFORMING REAL-LIFE PROBLEMS TO $AX=B$

Let's consider a Boundary Value Problem of the heat diffusion equation.

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad ; a<x<b$$

$$u(x,0) = u_0(x), \quad u(a,t)=A, \quad u(b,t) = B-e^{-t}$$

Steady-state problem is

Length of the bar

$$c^2 \frac{\partial^2 u}{\partial x^2} = 0, \qquad a<x<b$$

$$u(a) = A \qquad \text{or} \qquad u(a) = u_a$$

$$u(b) = B \qquad \text{or} \qquad u(b) = u_b$$

We generalize by considering some non-zero $f(x)$

So, we consider the below generalize system

$$\frac{\partial^2 u}{\partial x^2} = f(x) \qquad a<x<b$$

u at "a" $= u_a$

u at "b" $= u_b$

We can discretize the spatial domain (x) into a total of "n" sub-intervals. $\quad h = \dfrac{b-a}{n}$

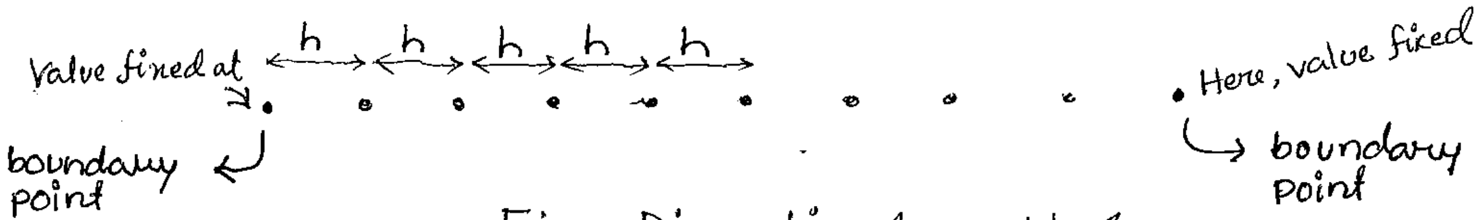

Fig. Discretized spatial Domain. (Mesh)

As the values are fixed at the two boundaries, values don't change. So, we treat the intermediate points as below:

$$\frac{\partial^2 u}{\partial x^2} = f(x)$$

$$\Rightarrow \quad \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} = f(x_i)$$ Using central Difference Method

$$\Rightarrow \quad \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} = \cancel{h} f(x_i)$$

Now, at mesh node $i = 1$

$$\frac{u_0 - 2u_1 + u_2}{h^2} = f(x_1)$$

$$\Rightarrow \quad \frac{u_a - 2u_1 + u_2}{h^2} = f(x_1)$$

u at "0" mesh is equal to u_a

@ mesh node $i = 2$

$$\frac{u_1 - 2u_2 + u_3}{h^2} = f(x_2)$$

@ mesh node $i = 3$.

$$\frac{u_2 - 2u_3 + u_4}{h^2} = f(x_3)$$

$\vdots$

@ mesh node $n-2$

$$\frac{u_{n-3} - 2u_{n-2} + u_{n-1}}{h^2} = f(x_{n-2})$$

@ mesh node $n-1$

$$(u_{n-2} - 2u_{n-1} + u_n)/h^2 = f(x_{n-1})$$

So, we obtain —

$$\frac{-2u_1 + u_2}{h^2} = f(x_1) - u_a/h^2$$

$$\frac{u_1 - 2u_2 + u_3}{h^2} = f(x_2)$$

$$\frac{u_2 - 2u_3 + u_4}{h^2} = f(x_3)$$

$$\vdots$$

$$\frac{u_{n-3} - 2u_{n-2} + u_{n-1}}{h^2} = f(x_{n-2})$$

$$\frac{u_{n-2} - 2u_{n-1} + u_n}{h^2} = f(x_{n-1})$$

We can write it in Matrix form:

$$\underbrace{\begin{bmatrix} -\frac{2}{h^2} & \frac{1}{h^2} & & & & \\ \frac{1}{h^2} & -\frac{2}{h^2} & \frac{1}{h^2} & & & \\ & \frac{1}{h^2} & -\frac{2}{h^2} & \frac{1}{h^2} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \frac{1}{h^2} & -\frac{2}{h^2} & \frac{1}{h^2} \\ & & & & \frac{1}{h^2} & -\frac{2}{h^2} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{n-2} \\ u_{n-1} \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} f(x_1) - u_a/h^2 \\ f(x_2) \\ f(x_3) \\ \vdots \\ f(x_{n-2}) \\ f(x_{n-1}) \end{bmatrix}}_{b}$$

$$AX = b$$

↳ Linear system representation

- Many real world problems are directly linear system

- There are other instances where the linear system occur as a part of the numerical analysis of other problems.
  ↳ this could be from the numerical analysis of many non-linear equations.

So, it is very important that a system of equations represented as

$$AX = B$$

is solved efficiently.

Information on the matrix "A"

The matrix "A" could be sparse or dense

Sparse : In many linear system, the matrix A contains a large number of zero' as its elements.

These matrices are known as sparse matrices

For example :

In many systems, the matrix A becomes tri-diagonal. as below :

$$A = \begin{bmatrix} a_1 & c_1 & 0 & \cdots & 0 \\ b_2 & a_2 & c_2 & 0 & \vdots \\ 0 & b_3 & a_3 & c_3 & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n & a_n \end{bmatrix}$$

diagonal 3

diagonal 2

diagonal 1

Special properties :

Often the matrix "A" has the special properties. For instance —

- It could be tridiagonal, banded
  - example given above
- It could be positive definite
  - That is

Given. $AX = B$

$$\Downarrow$$

$$\underbrace{X^T A X}_{} > 0$$

This is also called as quadratic form.

$AX = B$

vectors as matrix

All these properties are generally used to develop more efficient and computationally less expensive methods to solve

$$AX = B .$$

Solution Approach for $AX = b$

Direct Methods:

Given that arithmetic operations are done accurately, using some methods, the exact solution "x" of $AX = b$ can be calculated. These methods are known as Direct Methods.

↓ example

One of the most popular direct methods is the Gaussian Elimination.

↓ it reduces "A" to

Triangular form

↪ Then we do back substitution

Another example of direct method relies on the factorization of the matrix A as the product of two triangular matrices.

Here, factorization transforms $AX=b$ into two other easily invertable system.

$$
\begin{aligned}
& AX = b \\
\Rightarrow\ & B\underbrace{CX}_{y} = b \\
\Rightarrow\ & By = b \\
\text{and}\ & CX = y
\end{aligned}
\qquad \Bigg| \quad \text{Factorization } A = BC
$$

We invert the above systems and obtain X.

Iterative Methods :

Generally, these methods are applicable for all types of linear systems. However, they are commonly used for large sparse matrix.

often produced by discretizing partial differential equations.

LU Decomposition

Gaussian elimination with back-substitution shows that systems of equations involving triangular coefficient matrices (A) are easy to deal with.

Gaussian elimination : transforms A to a triangular matrix

From A to a upper triangular matrix.

Then, back-substitution

As the triangular matrices are easy to deal with,

LU decomposition transforms the matrix A into two different triangular matrices.

$$A = LU.$$

U → Upper triangular matrix

L → Lower triangular matrix

So, given
$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} U_{11} & & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

Lower triangular　　　　Upper triangular

So, Multiplying L and · obtain,

$$LU = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}U_{11} & L_{21}U_{12} + U_{22} & L_{21}U_{13} + U_{23} \\ L_{31}U_{11} & L_{31}U_{12} + L_{32}U_{22} & L_{31}U_{13} + L_{32}U_{23} + U_{33} \end{bmatrix}$$

|| equals to

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}$$

Element-wise comparison between LU & A provides:

$U_{11} = 1$, $U_{12} = 2$, $U_{13} = 4$

$L_{21}U_{11} = 3$ $\Rightarrow L_{21} = 3$ | $L_{21} + U_{22}$ $\Rightarrow U_{22} = 2$ | $L_{21} + U_{23} = 14$ $\Rightarrow U_{23} = 2$, | $L_{31}U_{11} = 2$ $\Rightarrow L_{31} = 2$

$L_{31}U_{12} + L_{32}U_{22} = 6$
$\Rightarrow 2 \times 2 + L_{32} \times 2 = 6$
$\Rightarrow L_{32} = 1$

$L_{31}U_{13} + L_{32}U_{23} + U_{33} = 13$
$\Rightarrow (2 \times 4) + (1 \times 2) + U_{33} = 13$
$\Rightarrow U_{33} = 3$

So, we obtain,

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} = \overset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}} \overset{U}{\begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}}$$

w Multiply LU to verify that it equals to A

Alternative Approach:

Given a matrix "A", we can transform "A" into an Upper triangular matrix by using Gauss elimination.

↳ Here, we perform elementary row operation.

For instance, Let's assume $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 6 & 8 \end{bmatrix}$

$$A = \begin{array}{c} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 6 & 8 \end{bmatrix} \end{array} \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 4 \end{bmatrix} \begin{array}{l} \\ R_2 \leftarrow R_2 - 2R_1 \equiv R_2 \leftarrow R_2 + (-2) R_1 \\ R_3 \leftarrow R_3 - 4R_1 \equiv R_3 \leftarrow R_3 + (-4) R_1 \end{array}$$

we make it zero

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{bmatrix} R_3 \leftarrow R_3 - 2R_2 \equiv R_3 \leftarrow R_3 + (-2) R_2$$

↳ This is upper-triangular matrix. U

Consider all the row-operations.

$$R_2 \leftarrow R_2 + (-2) R_1$$

$L_{21} \equiv$ inverse sign of (-2) opposite

$\equiv 2$

$$R_3 \leftarrow R_3 + (-4) R_1$$

$L_{31} \equiv$ inverse sign of (-4) opposite

$$R_3 \leftarrow R_3 + (-2) R_2$$

$L_{32} \equiv$ inverse sign of (-2)

So, Lower unit triangular matrix L

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

$$\Rightarrow L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix}$$

So,

$$\underbrace{A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 6 & 8 \end{bmatrix}}_{A} = LU = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{bmatrix}}_{U}$$

Using LU decomposition to solve systems of equations.

Given a linear system

$$AX = B$$

we write B. could write b also; notation only.

$$\Rightarrow \quad L\underbrace{UX}_{Y} = B \qquad \text{as } A = LU$$

$$\Rightarrow \quad LY = B$$

So, we solve

$$LY = B$$

Triangular system ① obtain Y

$$UX = Y$$

Triangular system ② obtain X

For instance: Given,

$$\underbrace{\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}}_{B}$$

Step 1: Calculate L & U according to the LU decomposition method. Here,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

Step 2: Solve $LY = B$ to obtain $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

so, we obtain,

$$LY = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix} = B$$

Now, we can use $\boxed{\text{forward substitution}}$ to obtain

$y_1 = 3$, $3y_1 + y_2 = 13$, and

Hence, $y_2 = 4$.

$2y_1 + y_2 + y_3 = 4$, and using $y_1 = 3$, and $y_2 = 4$, we obtain

$y_3 = -6$

Step 3: Now, we know $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix}$. So, we use it in $UX = Y$

so, $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix}$. We use <u>back-substitution</u>

We get $3x_3 = -6 \Rightarrow x_3 = -2$

$2x_2 + 2x_3 = 4$, $\Rightarrow x_2 = 4$, and

$x_1 + 2x_2 + 4x_3 = 3$, so, $x_1 = 3$

So, we found the solution of $AX = B$ using LU decomposition, and the solution is as follows:

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}; \text{ We found } X = \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}$$

□ Practice Problem:

$$\begin{bmatrix} 3 & 1 & 6 \\ -6 & 0 & -16 \\ 0 & 8 & -17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 17 \end{bmatrix}.$$

Find x_1, x_2, x_3