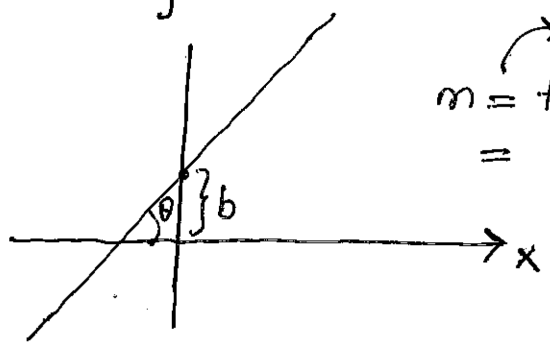
Tangent Line of a curve

- Tangent line touches a curve at one and only one point.
- Slope-intercept formula for a line
  $$\overset{\text{Slope}}{y = m}x + \underset{\text{y-intercept}}{b}$$
- Given a point (x_1, y_1), equation of the line passing through (x_1, y_1) is:

$$(y - y_1) = m(x - x_1)$$

$$\Rightarrow \quad \frac{(y - y_1)}{(x - x_1)} = m$$

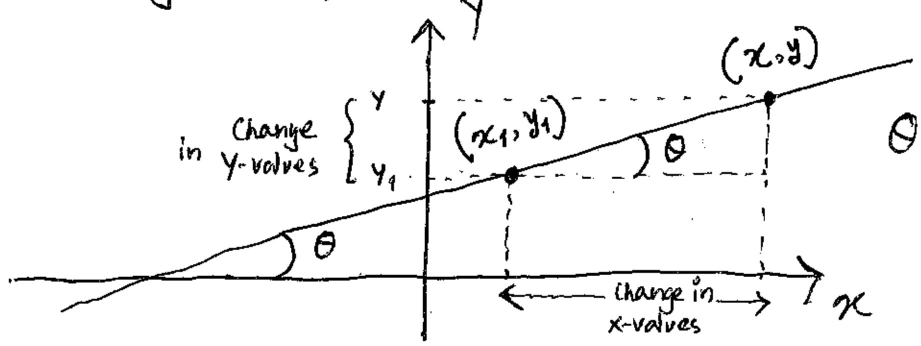
$$y = mx + b$$



$$\overset{\text{slope}}{m} = \tan\theta$$

$$= \frac{\text{Change in y-value}}{\text{Change in x-value}}$$

So, slope "m" for a line passing through two given points



$\theta \rightarrow$ Slope

So, $\tan\theta = \dfrac{\text{Perpendicular}}{\text{Base}} = \dfrac{\text{Change in y-values}}{\text{Change in x-values}}$

$$\Rightarrow \quad m = \frac{y - y_1}{x - x_1} \quad \Rightarrow \quad y - y_1 = m(x - x_1)$$

$$\Rightarrow \quad y = m(x - x_1) + y_1.$$

Slope and derivative

Given, a curve $f(x)$,

first derivative of $f(x)$ is the equation of the slope of the tangent line to the curve at any given point.

Definition of derivative :

$$\frac{d f(x)}{dx} \equiv \lim_{h \to 0} \frac{f(x+h) - f(h)}{h}$$

Example: Consider $f(x) = x^2$. Find the equation of the tangent line to $f(x) = x^2$ at $x = 2$'

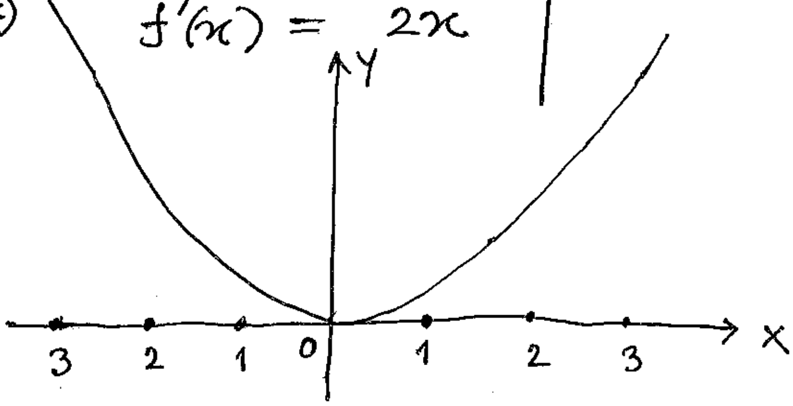
So,

$$f(x) = x^2$$

$$\Rightarrow f'(x) = 2x$$

$f'(2) = 2 \cdot 2 = 4$

↳ Slope of the tangent line



$f(x) = x^2$

$\Rightarrow f(2) = 4$

$\Rightarrow y = 4.$

So, $(2, 4)$ is the coordinate for the desired point.

Here, $(2,4)$ is the common point between the curve $f(x)$ and the tangent line.

Equation of the tangent line :

$$y - y_1 = m(x - x_1)$$

$$\Rightarrow y - 4 = 4(x - 2)$$

Find the tangent-line equation to:

$$f(x) = x^3 - 3x^2 + x - 1$$

at the given point $x = 3$.

Answer:

Step ①

$$f(3) = 3^3 - 3.3^2 + 3 - 1 = 27 - 27 + 3 - 1$$
$$= 2$$

So, the point where the tangent line will be drawn is $(3, 2)$.

Step ②

often termed as gradient

We find the slope of the tangent line:

$$f'(x) = 3x^2 - 6x + 1$$

so, $$f'(3) = 3.9 - 6.3 + 1 = 9 + 1 = 10$$

So, equation for tangent line

$$(y - y_1) = m(x - x_1)$$
$$\Rightarrow (y - 2) = 10(x - 3)$$
$$\Rightarrow y = 10x - 28$$

Ans

Taylor's Series

Consider a polynomial — say, $f(x) = a_0 + a_1 x + \dots a_n x^n$

Polynomial

x — involves only non-negative integers powers of x.

Generally, A $\boxed{\text{polynomial}}$ of degree "n" is a function that has the below form:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots \dots + a_n x^n$$

- real numbers
- known as coefficients

For instance,

$$f(x) = 4x^3 - 3x^2 + 2$$

- Polynomial of degree 3

$$f(x) = x^7 - 4x^5 + 1 \quad \text{degree ?}$$

Answer: 7

⊞ Degree of the polynomial means the highest order of x.

Highest Power of x	Degree of Polynomial	Name
x^0	0	constant
x^1	1	linear
x^2	2	quadratic
x^3	3	Cubic
x^4	4	quartic

Q. The fn^{c} $f(x)=0$, is it a polynomial?

Answer: Yes, it's a polynomial. But. it's degree is "undefined"

Roots of polynomials

Given a polynomial $f(x) = (x-a)(x-b)$, we can find the roots as below: $f(x) = 0$

$$\Rightarrow (x-a)(x-b) = 0$$

$$\Rightarrow x=a,\ x=b$$

Sometimes, roots could be repeated. For instance,

$$f(x) = (x-2)^2$$

Roots are, $f(x)=0 \Rightarrow (x-2)(x-2) = 0$

$$\Rightarrow x = 2, 2$$

Another polynomial, $f(x) = (x-2)^3 (x+4)^4$

So, roots are $\left[\begin{array}{l} x = 2, 2, 2 \\ x = -4, -4, -4, -4 \end{array}\right.$ Repeated roots

Here, root 2 has multiplicity 3

root -4 has multiplicity 4

Multiplicity of Roots:

If the multiplicity of roots is known, it provides information on the sketching of the given fn^{c}.

For instance, Let's say, we have a polynomial

$$f(x) = (x-2)^2(x+1)$$

Roots : 2, -1

Multiplicity 2

Here, maximum power of x is 3.

Shape of the graph

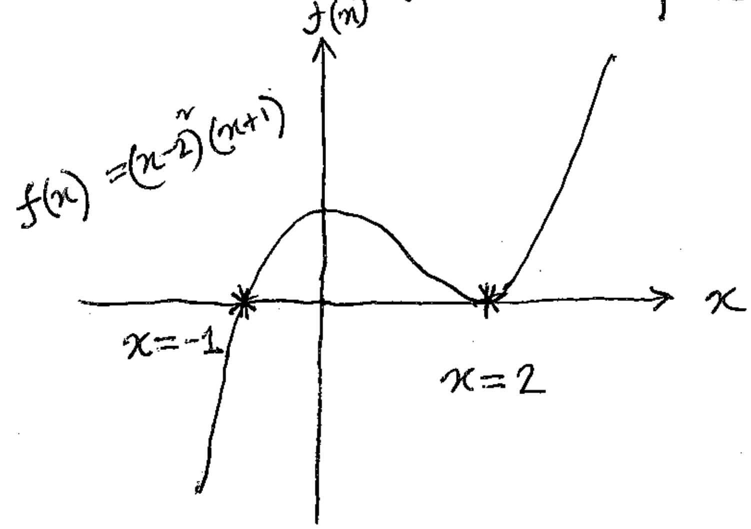
$$(x^2-4x+4)(x+1)$$
$$= (x^3-4x^2+\cancel{4x}+x^2-\cancel{4x}+4)$$
$$= (x^3-3x^2+4)$$

Positive coefficient

So, curve increases to right and decreases to left.

Because the root "2" has multiplicity 2 (which is even), the graph just <u>touches</u> x-axis.

The root "-1" has odd multiplicity. So, the graph crosses the x-axis.



So, multiplicity provides information whether a graph touches or intersects the corresponding axis.

Ex $f(x) = (x-3)^2(x+1)^5(x-2)^3(x+2)^4$

Root	Multiplicity	Touches/ Crosses x-axis
3	2	Touches
-1	5	Crosses
2	3	Crosses
-2	4	Touches

Power Series

Let's consider a special type of infinite series, namely the power series, which has the following form:

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

for real numbers a_n, and c. Here, x is the parameter.

A number of fn^c can be represented as power series: For instance,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n .$$

smooth fn^c

Generally, if a fn^c is infinitely differentiable, it can be represented as a power series of the form

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

$$= c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^2 + \cdots$$

$\Rightarrow f(a) = c_0$

Now, as $f(x)$ is differentiable for infinitely large times

so, $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots \cdots$

$\Rightarrow f'(a) = c_1$

$$f''(x) = 2c_2 + 3.2.c_3(x-a) + \dots\dots$$

$$\Rightarrow f(a) = 2c_2 \quad \Rightarrow \quad c_2 = \frac{f''(a)}{2} = \frac{f''(a)}{2!}$$

$$f'''(x) = 3.2.c_3 + 3.4.c_4(x-a)^2 + \dots\dots$$

$$\Rightarrow f'''(a) = 6.c_3 \Rightarrow \quad c_3 = f'''(a)/6 = \frac{f'''(a)}{3!}$$

Thus, we obtain, in general,

$$f^n(a) = n!\,c_n$$

$$\Rightarrow \quad c_n = \frac{f^n(a)}{n!}$$

Taylor series:

Let's assume that $f(x)$ has a power series expansion at $x=a$ with radious of convergence $\mathbb{R}>0$, then the series expansion of $f(x)$ has the below form:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + f'(a).(x-a) + \frac{f''(a).(x-a)^2}{2!} + f'''(a).\frac{(x-a)^3}{3!} + \dots\dots$$

When $a=0$, we get

Maclaurin Series

☐ Find Taylor's series for $f(x) = e^x$ at $x=0$

$$f(x) = f(a) + f'(a).(x-a) + \frac{f''(x-a)^2}{2!} + \frac{f'''(x-a)^3}{3!} + \dots \dots$$

Here, $f(x) = e^x$ | Here, $a=0$

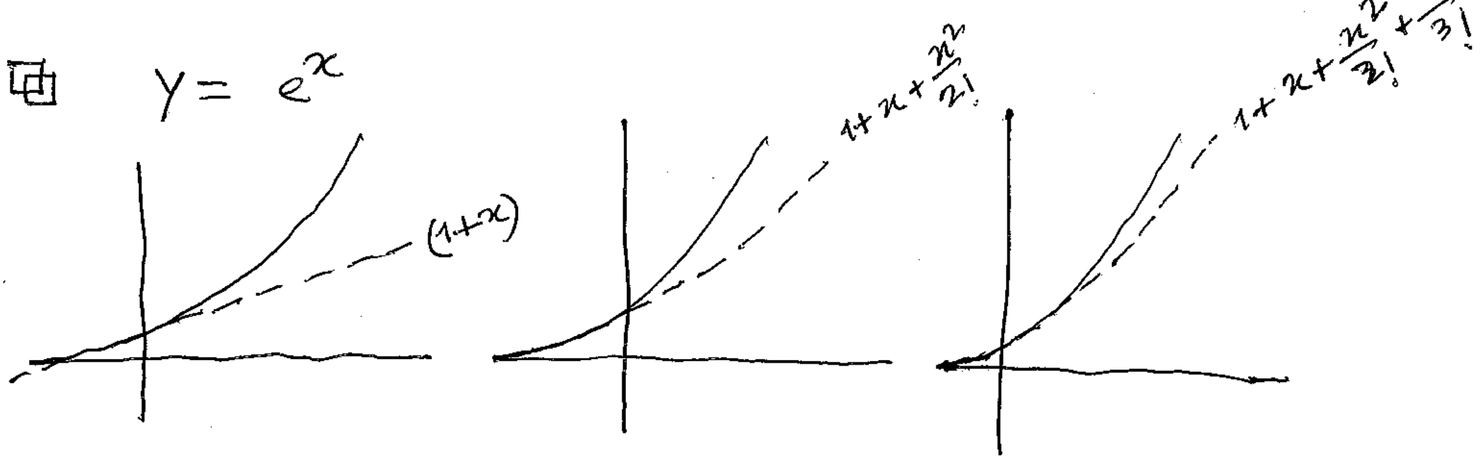
$\Rightarrow f(0) = e^0 = 1$

$f'(x) = e^x$ | $f''(x) = e^x$ | $f'''(x) = 0$

$\Rightarrow f'(0) = 1$ | $\Rightarrow f''(0) = 1$ | $\Rightarrow f'''(0) = 1$

So, $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \dots$$
$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

☐ $y = e^x$



$\longrightarrow$ error reducing w.r.t number of terms being considered.

☐ Find Taylor series of $\sin x = f(x)$
$\cos x = g(x)$.