

Solution & Necessary concepts.

Question 1: 6 + 8 Points

(a) What is the standard mathematical form of an Initial Value Problem (IVP)?

$$y'(t) = \frac{d}{dt}y(t) = f(t, y(t)), \quad a \leq t \leq b$$

$$y(a) = \alpha \text{ (alpha)}$$

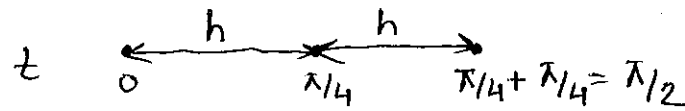
t : Independent variable

$y(t)$: Dependent variable

\hookrightarrow initial value

(b) Assume a differential equation $y'(t) = y^2 \sin(t)$ with values of $y(t)$ at $t = 0$ to be $y(0) = -3$. Use Euler method to approximate the value of $y(t)$ at $t = \pi/2$ by performing at least 2 iteration steps.

As we need at least two iterations, h could be $\pi/4$. see graphically



So, $y(0) = -3$ &
 $h = \pi/4$
 Here, $f(t, y(t)) = y^2 \sin(t)$

From Euler Method:

iteration 1: $y_1 \xrightarrow{\text{at } \pi/4} = y_0 + h f(t_0, y_0) \xrightarrow{=0} = -3 \xrightarrow{=-3}$

$$= -3 + \pi/4 \cdot (-3)^2 \sin(0) = -3 + 0 = -3$$

iteration 2: $y_2 \xrightarrow{\text{at } \pi/2} = -3 + \pi/4 (-3)^2 \sin(\pi/4) = -3 + 4.9982$

$$\approx 1.9982 \quad \underline{\text{Ans}}$$

Here, $y_2 \equiv y(t)$ after two iterations,
 \hookrightarrow takes us to $\pi/4 + \pi/4 = \pi/2$

Question 1: 8 Points

Given a function $f(x, y) = |y|$, check if it satisfies Lipschitz condition. If it satisfies, find out the Lipschitz constant (L). Hints: You might find the formula of Triangle Inequality $|a+b| \leq |a|+|b|$ useful for this problem.

According to Lipschitz condition

Now, given $f(x, y) = |y|$

$$\text{so, } f(x, y_1) = |y_1|$$

$$f(x, y_2) = |y_2|$$

$$\Rightarrow f(x, y_1) - f(x, y_2) = |y_1| - |y_2|$$

$$\Rightarrow |f(x, y_1) - f(x, y_2)| = \underbrace{||y_1| - |y_2||}$$

we can write:

$$y_1 = y_1 - y_2 + y_2$$

$$\Rightarrow |y_1| = \left| \frac{y_2}{a} + \frac{y_1 - y_2}{b} \right|$$

$$\Rightarrow |y_1| \leq |y_2| + |y_1 - y_2| \quad \text{applying the Triangle Inequality}$$

$$\Rightarrow |y_1| - |y_2| \leq |y_1 - y_2| \quad \dots \textcircled{1}$$

$$y_2 = y_1 - y_1 + y_2$$

$$\Rightarrow y_2 = y_1 + y_2 - y_1$$

$$\Rightarrow |y_2| = |y_1 + y_2 - y_1|$$

$$\leq |y_1| + |y_2 - y_1|$$

$$\leq |y_1| + |y_1 - y_2|$$

$$\Rightarrow -|y_1 - y_2| \leq |y_1| - |y_2| \quad \dots \textcircled{2}$$

From $\textcircled{1}$ and $\textcircled{2}$, we can write

$$-|y_1 - y_2| \leq |y_1| - |y_2| \leq |y_1 - y_2|$$

Definition of

$$\Rightarrow ||y_1| - |y_2|| \leq |y_1 - y_2|$$

Lipschitz constant.

$$\text{so, } |f(x, y_1) - f(x, y_2)| \leq |y_1 - y_2| \equiv 1 \cdot |y_1 - y_2|$$

That is, $f(x, y) = |y|$ satisfies Lipschitz condition.

A fn^s $f(x, y)$ satisfies a Lipschitz condition in y on set $D \subset \mathbb{R}^2$ if there exists a constant $L > 0$ such that $|f(x, y_1) - f(x, y_2)| \leq L |y_1 - y_2|$

we need to replace it with $|y_1 - y_2|$, and also, we need to obtain \leq