Fixed Point Iteration Schemes

While the simple enclosure method is guranteed to converge to a root of equation,
it often suffers from slow rate of convergence

To overcome the slow rate of convergence, we have options of Fixed-point Iterations

$\downarrow$ When properly constructed

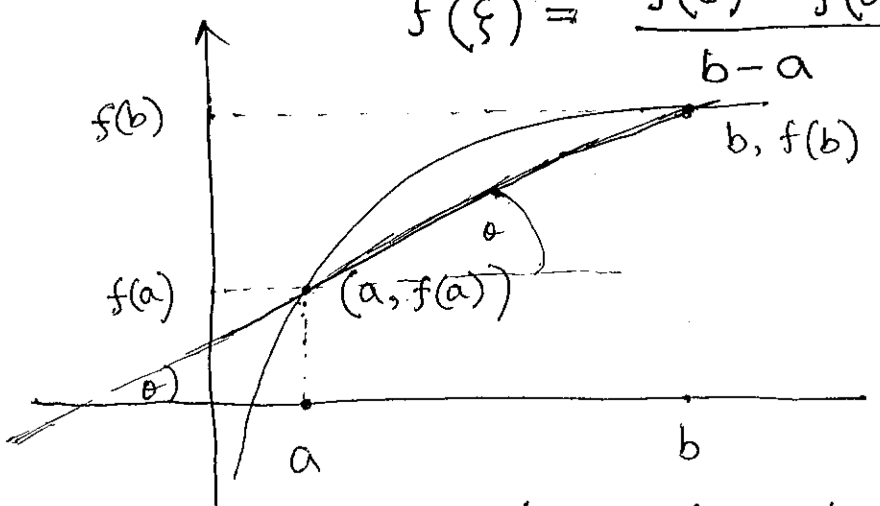
Provides rapid convergence.

But, at the cost of the loss of guranteed convergence.

Review of Mean Value Theorem:

Theorem: If the function "f" is continuous on the close interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a real number $\xi \in (a, b)$ such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}$$

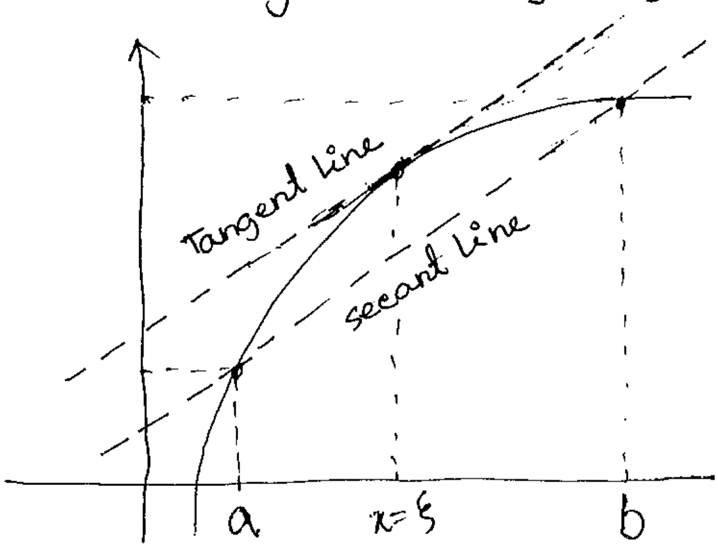


Slope: $\theta = \dfrac{y_2 - y_1}{x_2 - x_1}$

So, $f'(\xi)$ gives the slope of a line passing through $(a, f(a))$ and $(b, f(b))$

☐ Line passing through this is often called secant line.

$f'(\xi)$ gives the slope of the line tangent to fn^c f at $x = \xi$.



☐ So, the Mean Value Theorem gurantees —

That there's at least one point on the graph of the fn^c f at which the tangent line is parallel to the secant line. Eq. goes as follows:

$$f'(\xi) = \frac{f(b) - f(a)}{(b-a)}$$

$$\Rightarrow \quad f'(\xi)(b-a) = f(b) - f(a)$$

$$\Rightarrow \quad f(b) - f(a) = f'(\xi)(b-a)$$

$\downarrow$

This form relates difference of two fn^c values to the difference of the corresponding input values.

Fixed Points : BACKGROUND

A fixed point of the function 'g' is any real number, p, for which $g(p) = p$; that is, whose location is fixed by g

Example: Consider $g(x) = \sin x$ as the function.

$$\sin(\pi/4) = \frac{1}{\sqrt{2}}$$

$$\sin(0) = 0$$

Here, Sine fn^c maps "0" to "0"; sine fn^c fixes the location of "0"

Hence, $x = 0$ is a fixed point of the fn^c $\sin x$.

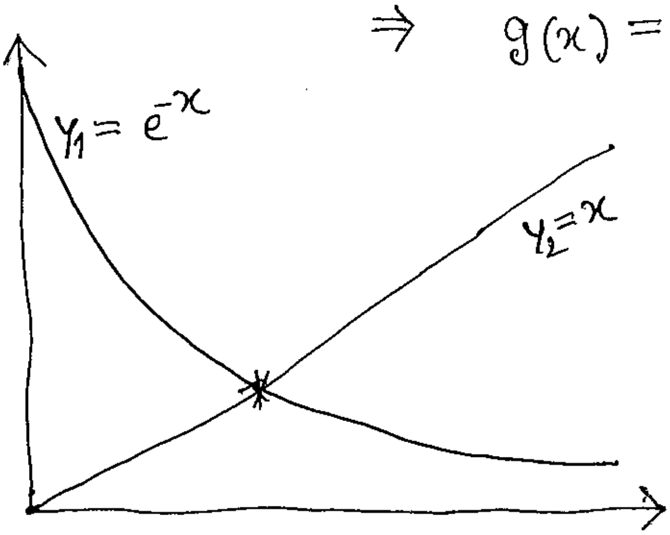
When analytical approach fails, we can use graphical approach — Let's consider $y_1 = g(x)$; $y_2 = x$

↳ Same set of coordinates

So, if the two graphs intersect, we obtain

$$y_1 = y_2$$

$$\Rightarrow g(x) = x$$



Assume $y_1 = g(x) = e^{-x}$

$y_2 = x$

So, intersecting point provides $e^{-x} = x$ at that point.

↓

Fixed point

- When analytical approach fails, we can use graphical approach to investigate the existence of fixed points.

Let's assume $y_1 = g(x)$ $y_2 = x$. Intersection provides

$$y_1 = y_2 \Rightarrow g(x) = x$$

Now, if $g(x) = e^{-x}$, then $x = e^{-x}$.

That's we obtain the fixed point.

- issues to consider ...
  - fn^c can have two fixed points
  - fn^c can have unique fixed point
  - fn^c may not have any fixed point
    - ↳ x^2+1 has no fixed point.

- Conditions under which a fn^c is guaranteed to have a unique fixed point.

Theorem :

existence: Let g be continuous on interval $[a, b]$;

$$g : [a,b] \rightarrow [a,b]$$

Then g has a fixed point $P \in [a,b]$.

Furthermore,

Uniqueness: if g is differentiable on the open interval (a,b) and there exists a positive constant $K<1$, such that $|g'(x)| \leq K < 1$ for all $x \in (a,b)$

then the fixed point in $[a,b]$ is unique.

Existence

g is continuous on the interval $[a,b]$ with

$$g : [a,b] \to [a,b]$$

Define.

$$h(x) = g(x) - x$$

→ continuous on interval $[a,b]$
→ " " " $[a,b]$

↓ this implies

PROPERTIES ①

h(x) is also continuous

PROPERTIES ②

By construction roots of $h(x)$ are precisely the fixed point of g

$$\underbrace{h(x)=0}_{\text{roots}} \Rightarrow g(x) - x = 0 \Rightarrow \underbrace{x = g(x)}_{\text{fixed points}}$$

Now, since
$$\min_{x\in[a,b]} g(x) \geq a$$
$$\max_{x\in[a,b]} g(x) \leq b$$

we obtain,

$$h(a) = g(a) - a \geq 0 \text{ , and}$$
$$h(b) = g(b) - b \leq 0$$

Now, if $h(a)$ or $h(b) = 0$, then we have a root.

If not, (neither $h(a)=0$, nor $h(b)=0$)

because of this
$\leq$ becomes $<$

We can say $h(b) < 0 < h(a)$

Since h is continuous on $[a,b]$, the intermediate value theorem may be invoked.

$P \in [a,b]$

such that $h(P) = 0$

$\Rightarrow P = g(P)$

Uniqueness...

Let's assume that p and q are both fixed points, of the fn^c g on the interval $[a, b]$.

Here, $p \neq q$. so, $g(p) = p$, $g(q) = q$

Then, $|p-q| = |g(p) - g(q)|$

$= |g'(\xi)(p-q)|$ using mean value theorem.

$= |g'(\xi)||p-q|$

$\leq K|p-q| < |p-q|$

As $|p-q| < |p-q|$ with p, q being the two different fixed points, with theorem mentioned earlier, this is a contradiction.

So, Fixed point is unique.

Comments:

Hypotheses made in this theorem are sufficient conditions to guarantee the existence and uniqueness of a fixed point.

↳ these are not necessary conditions.

↓

A fn^c can violate one or more of the hypotheses, but still have a unique fixed point.

Fixed point Iteration

If a fn^c g has a fixed point, one way to approximate the value of fixed point is to use fixed point iteration scheme.

Given a starting approximation "p_0"

A Fixed Point Iteration scheme to approximate the fixed point, p, of a function g, generates the sequence $\{p_n\}$ by the rule

$$p_n = g(p_{n-1}) \quad \text{for all } n \geqslant 1.$$

How is it related to rootfinding problem?

In principle, every rootfinding problem can be transformed into different fixed point problems.

- Some will converge rapidly
- Some will converge slowly
- Some will not converge at all.

So, rootfinding equation

$$f(x) = 0$$

$$\Big\downarrow \text{ Transformed algebraically}$$

$x = \dots\ ; \Rightarrow x = g(x)$

Consider $f(x) = x^3 + x^2 - 3x - 3$

$$\Rightarrow x^3 + x^2 - 3x - 3 = 0$$

$$\Rightarrow x = \frac{1}{3}(x^3 + x^2 - 3)$$

Alternatively,
$$g_2(x) = -1 + \frac{3x+3}{x^2}$$
$$g_3(x) = \sqrt[3]{3+3x-x^2}$$
$$g_4(x) = \sqrt{(3+3x-x^2)/x}$$
$$g_5(x) = x - (x^3+x^2-3x-3)/(3x^2+2x-3)$$

We consider a start point $P_0 = 1$, and use a stopping criteria $|P_n - P_{n-1}| < 10^{-7}$.

That is,

We reformulate the rootfinding equation $f(x)=0$ to an equivalent fixed point problem.

$$f(x)=0 \iff g(x) = x$$

With an initial guess x_0

We compute the sequence.

$$x_{n+1} = g(x_n)$$

n, the iteration

we hope that

$x_n \to P$

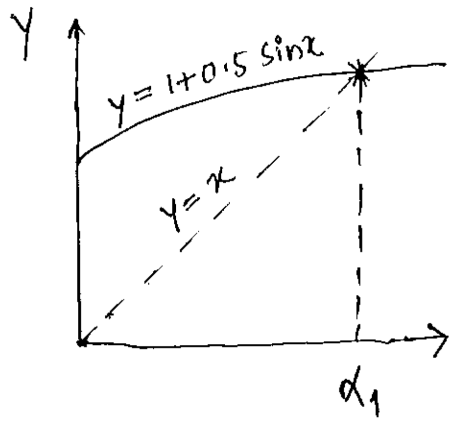
root

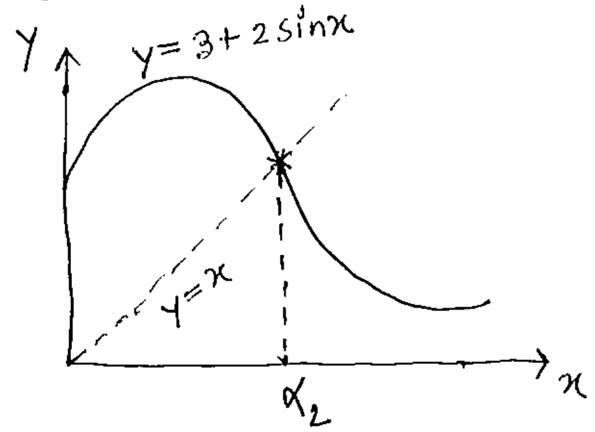
✓ There are infinite many ways to introduce an equivalent fixed point problem, for a given equation.

Consider examples of two equations:

$$x = 1 + 0.5 \sin x \quad \cdots \cdots (1)$$

$$x = 3 + 2 \sin x \quad \cdots \cdots (2)$$





Precisely, $\alpha_1 = 1.49870113351785$

$\alpha_2 = 3.09438341304928$

With initial guess x_0 $\downarrow$ we obtain $x_1 \longrightarrow x_2$

* this process is repeated until the convergence occurs.

Here, convergence does not occur for eq^n (2)

But why is it not converging ?

Based on the results that for some $g(x)$, iteration converge
for some $g(x)$ do not

AND

for some $g(x)$ iteration converges, but to a fixed point outside the given interval.

So, we must know —

Given that a fn^c. f, can an iteration fn^c g, be constructed such that fixed points of g are the roots of "f"

AND

for some starting point/approximation P_0, the sequence $P_n = g(P_{n-1})$ converges to P.
P is the root of "f"

To answer this :

We need to know the conditions on "g" that will guarantee convergence of iteration.

Theorem:
Let's consider "g" as continuous on the closed interval $[a, b]$

with $g : [a, b] \rightarrow [a, b]$

Furthermore, Let's assume "g" is differentiable on the open interval $(a\ b)$.

and

there exists a positive constant $K<1$ such that $|g'(x)| \leqslant K < 1$; for all $x \in (a,b)$

Then,

1. Seqvence $\{P_n\}$ generated by $P_n = g(P_{n-1})$ converges to the fixed point p for any $P_0 \in [a,b]$

2. $|P_n - P| \leqslant K^n \max(P_0 - a, b - P_0)$

3. $|P_n - P| \leqslant \frac{K^n}{1-K} |P_1 - P_0|$

⌗ We recall those two examples again

$$x = 1 + 0.5 \sin x \quad \dots \quad \dots \quad (1)$$

$$x = 3 + 2 \sin x \quad \dots \quad \dots \quad (2)$$

Here,

$g(x) = 1 + 0.5 \sin x$	$g(x) = 3 + 2\sin x$
$\Rightarrow g'(x) = 0.5 \cos x$	$\Rightarrow g'(x) = 2\cos x$
For, $\alpha = 1.4987$	For $\alpha = 3.094$
So, $\lvert g'(\alpha)\rvert \leq \frac{1}{2}$	$g'(\alpha) = -1.998$
↳ condition satisfied ↓ Convergence occur	$\Rightarrow \lvert g'(\alpha)\rvert = 1.998$ ↳ Violating the condition.

Hypotheses of this theorem are precisely those that were sufficient to guarantee that a function has unique fixed point.

Proof :

(1) To establish the first part of the theorem, we must show that

$$\underbrace{|P_n - P|}_{\text{errore}} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty$$

So,
$$|P_n - P| = |g(P_{n-1}) - g(P)|$$
$$= |g'(\xi)||P_{n-1} - P| \quad \text{Mean value theorem}$$
$$\leq K|P_{n-1} - P| \quad \text{Hypothesis on } g'$$

$$|P_{n-1} - P| = |g(P_{n-2}) - g(P)|$$
$$= |g'(\xi)||P_{n-2} - P|$$
$$\leq K|P_{n-2} - P|$$

That is,
$$|P_n - P| \leq K|P_{n-1} - P|$$
$$\leq K^2 |P_{n-2} - P|$$
$$\vdots$$
$$K^n |P_0 - P|$$

since $K < 1$

$$\lim_{n\to\infty} |P_n - P| \leq \lim_{n\to\infty} K^n |P_0 - P| = |P_0 - P| \lim_{n\to\infty} K^n$$
$$= 0$$

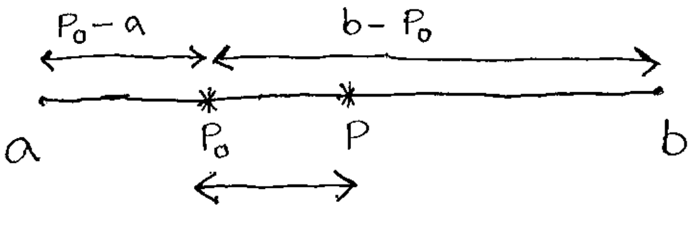
2 We obtained

$$|P_n - P| \leq K^n |P_0 - P|$$
$$\leq K^n \max\{P_0 - a,\ b - P_0\}$$
since $P \in [a, b]$

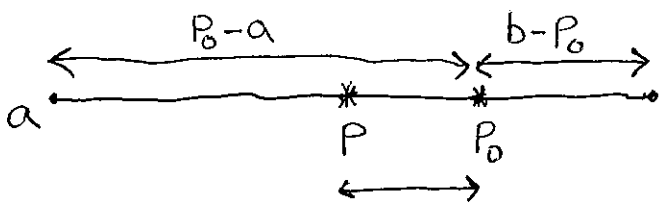
To simplify the proof –

Error formula $|P_n - P| \leq K^n |P_0 - P|$

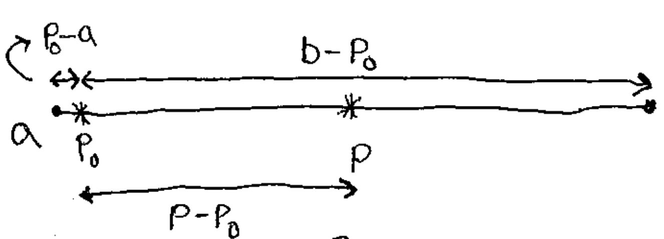
Considering
- P : As the fixed point
- P_0 : Initial approximation
- a : Left bound of interval
- b : Right bound of interval



$\max\{P_0 - a,\ b - P_0\}$
$= b - P_0$



$\max\{P_0 - a,\ b - P_0\}$
$= P_0 - a$



$\max\{P_0 - a,\ b - P_0\}$
$= b - P_0$

Whatever the example of P_0 chosen, $\max\{P_0 - a,\ b - P_0\}$ is always greater than $|P_0 - P|$.

Using the information as obtained graphically, we can write

$$|P_0 - P| < \max\{P_0 - a,\ b - P_0\}$$

By using $|P_0 - P|$ inequality in the error formula

$$\cancel{|P_n - P| < \max\{P_0 - a,\ b - P_0\}}$$

$$\Rightarrow |P_n - P| \leqslant K^n \underbrace{|P_0 - P|}_{\leqslant \max\{P_0 - a,\ b - P_0\}}$$

$$\Rightarrow |P_n - P| \leqslant \max\{P_0 - a,\ b - P_0\}$$

Proved

Proof for: $|P_n - P| \leqslant \frac{K^n}{1-K} |P_1 - P_0|$

$|P_{n+1} - P_n| = |g(P_n) - g(P_{n-1})|$ Definition of P_{n+1} & P_n

$= |g'(\xi)|\,|P_n - P_{n-1}|$ Using Mean Value Thr^m

$\leqslant K|P_n - P_{n-1}|$ Hypothesis on g'

$\leqslant K^2|P_{n-1} - P_{n-2}|$

$\leqslant K^3|P_{n-2} - P_{n-3}|$

$\vdots$

$\leqslant K^n|P_{n-(n-1)} - P_{n-n}|$

$= K^n|P_1 - P_0|$

So, $|P_{n+1} - P_n| \leqslant K^n|P_1 - P_0|$

$^{W}|P_n - P_{n-1}| = |g(P_{n-1}) - g(P_{n-2})|$

$= |g'(\xi)|\,|P_{n-1} - P_{n-2}|$

$\leqslant K|P_{n-1} - P_{n-2}|$

$|P_{n-1} - P_{n-2}| = |g(P_{n-2}) - g(P_{n-3})|$

$= |g'(\xi)|\,|P_{n-2} - P_{n-3}|$

$\leqslant K|P_{n-2} - P_{n-3}|$

So, we obtained

$$|P_{n+1} - P_n| \leqslant K^n |P_1 - P_0| \quad \cdots \text{(1)}$$

Assuming $m > n$, we can write.

$$P_m - P_n = |P_m - P_{m-1} + P_{m+1} - P_{m-2} + P_{m+2} + \cdots P_{n+1} - P_n|$$

$$\leqslant \underbrace{|P_m - P_{m-1}| + |P_{m-1} - P_{m-2}| + \cdots \cdots + |P_{n+1} - P_n|}$$

we are generalizing for any $m > n$
In (1) we got for $n+1 > n$. We generalize by assuming $m > n$

$$\leqslant \underbrace{K^{m-1}|P_1 - P_0| + K^{m-2}|P_1 - P_0| + \cdots + K^n |P_1 - P_0|}$$

$$= K^n |P_1 - P_0| \left(1 + K + K^2 + \cdots + K^{m-n-1}\right)$$

Now, as $m \to \infty$, approximated fixed point P_m converges towards true fixed point P.

Thus, we can write—

$$|P_m - P_n| = |P - P_n| \leqslant K^n |P_1 - P_0| \left(1 + K + K^2 + \cdots + K^\infty\right)$$

$$\leqslant K^n |P_1 - P_0| \underbrace{\left(1 + K + K^2 + \cdots \cdots\right)}_{= \frac{1}{1-K}}$$

$$\leqslant K^n |P_1 - P_0| \cdot \frac{1}{1-K}$$

Here, as K is smaller convergence occur faster. As $K \to 1$, rate of convergence significantly.

When $K = \frac{1}{2}$, rate of convergence of fixed point iteration scheme becomes almost equal to the rate of convergence of Bisection method.

NEWTON'S METHOD

- Often provides faster convergence than the other types of functional iteration.

Let's assume that $f \in C^2[a,b]$, and $P_0 \in [a,b]$ is an approximation to p such that

$f'(P_0) \neq 0$, and $|P - P_0|$ is small

We consider a Taylor expansion of f about P_0 and evaluated at $x = P$.

$$f(P) = f(P_0) + \frac{f'(P_0)}{1!}(P-P_0) + \frac{f''(P_0)}{2!}(P-P_0)^2 + \text{Higher orders}$$

$$\Rightarrow f(P) = f(P_0) + f'(P_0)(P-P_0)$$

as $(P-P_0)$ is small, $(P-P_0)^2$ is negligible

$$\frac{f''(P_0)(P-P_0)^2}{2!} = \frac{(P-P_0)^2}{2!} f''(\xi(P))$$

Lagrange form of the remainder of Taylor series expansion

$\Rightarrow f(P) = 0$ as P is the root.

Now,

$$f(P_0) + f'(P_0)(P-P_0) = 0$$

$$\Rightarrow (P-P_0) = -\frac{f(P_0)}{f'(P_0)}$$

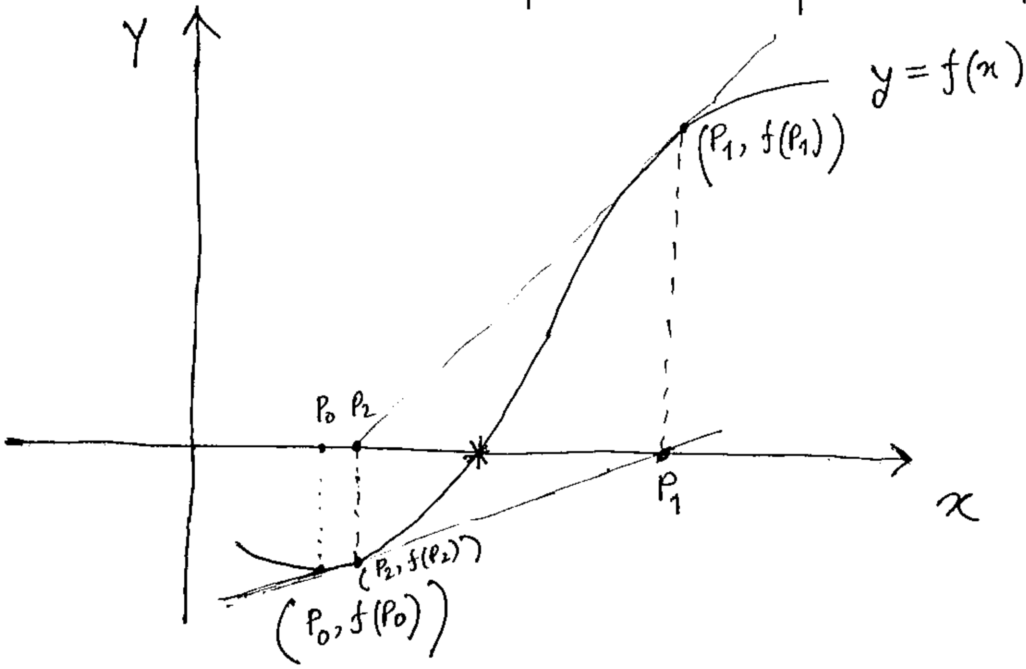
$$\Rightarrow P = P_0 - \frac{f(P_0)}{f'(P_0)}$$

$$\Rightarrow P_1 = P_0 - \frac{f(P_0)}{f'(P_0)}$$

So, with initial approximation P_0, Newton's method generates the sequence

$$P_n = P_{n-1} - \frac{f(P_{n-1})}{f'(P_{n-1})}, \quad n \geq 1$$

Newton's Method : Graphical representation



- Starting from P_0, approximation P_1 is the x-intercept of the tangent line to $f(x)$ at $(P_0, f(P_0))$.
- Approximation P_2 is the x-intercept of the tangent line of $f(x)$ is drawn at $(P_1, f(P_1))$.

and so on

We can use "x" instead of "P" to represent in the form

$$g(x) = x - \frac{f(x)}{f'(x)}$$

Let's say, P_0 is the initial approximation.

So, we generate the sequence $\{P_n\}$ using $P_n = g(P_{n-1})$

$\Rightarrow P_{n+1} = g(P_n)$.

- Newton's method requires two fn^c evaluation : $f'(x)$ & $f(x)$ in each iteration

whereas, in Bisection method & False position method, we need one fn^c evaluation.