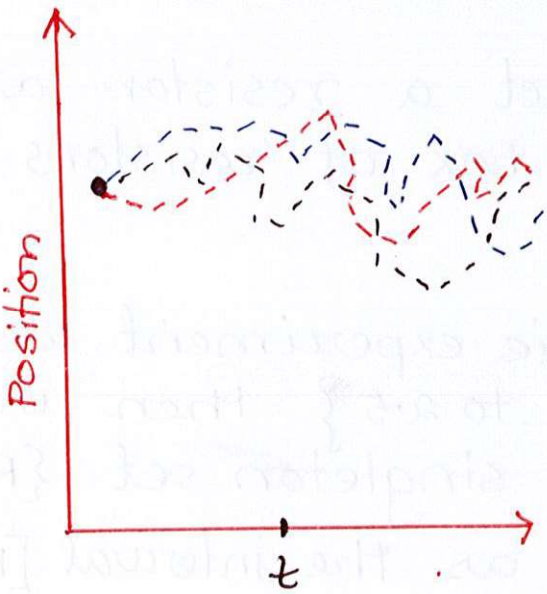
We can ask a fundamental question

Why do we describe a random variable (RV) as a mapping in the first place?

- Insight on physical problems, such as induced current on a placed antenna because of thermal movement of charges at any given instance of time
- Secondly, we can define multiple random variables on a random experiment
- In fact, we can generalize the simple random variables to random processes or stochastic processes.

Sequence of RVs at a given time point when the experiment involves dynamic evolution in time. For instance, observing a particle moving randomly in time.



If the position is tracked and multiple observations are made, position of the particle at a given time point will be a sequence of RVs.

A few examples:

- Let $(S, \mathcal{F}, P)$ be a probability space and assume that event $A \in \mathcal{F}$. We can define random variable
$$X(\omega) = 1_A(\omega)$$

- Again, suppose a die has six colored side and the die is rolled as the random experiment. So,
$$S = \{\text{Red, Green, Yellow, Blue, Cyan, Magenta}\}$$
$$\mathcal{F} = \mathcal{P}(S)$$
We can define a random variable (RV) as follows:
$$X(R) = 1, \; X(G) = 2, \; X(Y) = 3, \; X(B) = 4,$$
$$X(C) = 5, \; X(M) = 6$$
measurable RV

- Let's say we select a resistor at random from a box of resistors and measure its value.

For instance, for the die experiment we choose a Borel set $\{1.5 \text{ to } 2.5\}$ then in pre-image we will have singleton set $\{R\}$
If the Borel set is taken as the interval $[1.5$
Then, pre-image of that would be
$\{\text{Red, Green}\}$, which will be in the power set

That is, measurable

So, the discussion shows that

we can think of a random variable in two ways:

Mathematically, a random variable is a measurable function from $(S, \mathcal{F}, P)$ to $\mathbb{R}$.

precise mathematical way to think about a RV

Intuitively, Something that takes on real values at random.

The outcomes of a random experiment, where real values are generated at random $(\mathcal{C}, \mathcal{B}(\mathcal{C}), P)$, where $\mathcal{C} \subset \mathbb{R}$

set of all possible values that can be taken of at random

Given a random variable with range space $\mathcal{C} \subset \mathbb{R}$, we can construct a probability space $(\mathcal{C}, \mathcal{B}(\mathcal{C}), P_X)$.

Then, the question is —

How do we describe P_X?

Two cases
- Discrete
- Continuous

Discrete case: For P_X, we can use the probability mass function (pmf), denoted as

$$P_X(x) = P_X(\{x\}), \forall x \in \mathcal{C}$$

with properties: (i) $P_X(x) \geq 0, \forall x \in \mathcal{C}$

(ii) $\sum_{x \in \mathcal{C}} P_X(x) = 1$

Continuous case: For continuous case, we use the probability density function (pdf) to assign

$$P_X(G) = \int_G f_X(x)\,dx, \quad \forall G \in B(\mathcal{C})$$

with properties as: (i) $f_X(x) \geq 0, \forall x \in \mathcal{C}$

(ii) $\int_{\mathcal{C}} f_X(x)\,dx = 1$

continuous case is the uncountable sample space

However, even in the discrete case we have enormously large number of events and assigning probabilities to all those are not simple task. To make the process easier different mathematical tools are used and one such tool is the

Cumulative Distribution Function, shortly known as CDF

Definition of CDF

Given a random variable X defined on $(S, \mathcal{F}, P)$ that includes a new probability space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), P_X)$, the cumulative distribution function "cdf" of X is defined as

$$F_X(\alpha) = P_X((-\infty, \alpha]) = P_X(\{x : x \leq \alpha\})$$

alpha

$$= P(\{\omega \in S \mid X(\omega) \leq \alpha\})$$

$$= P(X^{-1}((-\infty, \alpha])), \quad \alpha \in \mathbb{R}$$

As observed, the cdf notion accomodates alternative options.

For instance, $\underbrace{\{X \leq \alpha\}}_{\text{event}}$ or $P(\{X \leq \alpha\})$

represents

$$\underbrace{\{\omega \in S \mid X(\omega) \leq \alpha\}}_{\text{set of all } \omega \text{ in the sample space as defined in } (S, \mathcal{F}, P)} \in \mathcal{F}$$

□ Essence of $F_X(\alpha)$: If we know $F_X(\alpha)$ it completely specifies the probability measure $P_X(\cdot)$

We know that. It is possible to construct any set $F \in \mathcal{B}(\mathbb{R})$, where F acts as the event, using countable sequence of set operations on the intervals of the form $(-\infty, x_n]$.

Although sen-close semi-open intervals are considered, it is actually possible. Because,

we already have seen that any event F in the Borel field by performing a countable sequence of set operations on the open interval of the form (α_n, β_n)

we saw that it is not hard to get intervals of the form $(-\infty, x_n)$ from (α_n, β_n) types of intervals.

For instance,

$$[a, b) = \{a\} \cup (a, b) \in B(\mathbb{R})$$

$$(a, b] = (a, b) \cup \{b\} \in B(\mathbb{R})$$

$$[a, b] = \{a\} \cup (a, b) \cup \{b\} \in B(\mathbb{R})$$

So, we write $F_X(x_n) = P_X((-\infty, x_n])$

That is, starting from cdf $F_X(\alpha)$, it is possible to get $P_X(\cdot)$ for any set (event) we are interested in.

means that CDF is a complete probabilistic description of any random experiment

So, the cdf is defined as:

$$F_X(x) = P(\underbrace{\{X \leq x\}}_{\text{event stating that } X \leq x}), \quad \forall x \in \mathbb{R}$$

$$= P_X((-\infty, x]), \quad \forall x \in \mathbb{R}$$

$$= P(\{\omega \in S \mid X(\omega) \leq x\}), \quad \forall x \in \mathbb{R}$$

Properties of cdf:

1. $F_X(+\infty) = 1$ and $F_X(-\infty) = 0$

Explanation: $F_X(+\infty) = P(\underbrace{\{X \leq +\infty\}}_{\text{becomes } S})$

$$= P(S) = 1$$

$$F_X(-\infty) = P(\underbrace{\{X \leq -\infty\}}_{\text{impossible event } \emptyset})$$

$$= P(\emptyset) = 0$$

2. If $x_1 < x_2$, then $F_X(x_1) \leq F_X(x_2)$

Explanation: $F_X(x_1) = P_X(-\infty, x_1])$

$F_X(x_2) = P_X((-\infty, x_2])$

Observing the real line, we can write

$$(-\infty, x_1] \subset (-\infty, x_2]$$

we can write $(-\infty, x_2]$ as the union of two disjoint sets

$$(-\infty, x_2] = (-\infty, x_1] \cup (x_1, x_2]$$

$$\Rightarrow P_X((-\infty, x_2]) = P_X\Big((-\infty, x_1] \cup (x_1, x_2]\Big)$$

$$= P_X((-\infty, x_1]) + P_X((x_1, x_2])$$

$$\Rightarrow F_X(x_2) = F_X(x_1) + \underbrace{P_X((x_1, x_2])}_{\text{which is greater than zero}}$$

$$\Rightarrow F_X(x_2) \geq F_X(x_1)$$

3. $P(\{X > \alpha\}) = 1 - F_X(\ )$

The event $\{X > \alpha\} \equiv \overline{\{\ \leq \alpha\}}$

As, $\{X > \alpha\}$ and $\{X < \alpha\}$ are disjoint their union

$$\{X > \alpha\} \cup \{X \leq \alpha\} = S$$

$$\Rightarrow P(\{X > \alpha\}) + P(\{X \leq \alpha\}) = P(S)$$

$$\Rightarrow P(\{X > \alpha\}) = 1 - P(\{X \leq \alpha\})$$

$$\Rightarrow P(\{X > \alpha\}) = 1 - F_X(\alpha)$$

4. If $x_1 < x_2$, then $P(\{x_1 < \ \leq x_2\}) = F_X(x_2) - F_X(x_1)$

We can write $(-\infty, x_2]$ as the union of two disjoint set

$$(-\infty, x_2] = (-\infty, x_1] \cup (x_1, x_2]$$

$$\Rightarrow P(\{-\infty < X \leq x_2\}) = P\Big(\{-\infty < X \leq x_1\} \cup \{x_1 < X \leq x_2\}\Big)$$

Using the definition, we can write

$$F_{X}(x_2) = F_{X}(x_1) + P(\{x_1 < X \leq x_2\})$$

$$\Rightarrow P(\{x_1 < X \leq x_2\}) = F_{X}(x_2) - F_{X}(x_1)$$

Proved

5. Right continuity of CDF

If $F_X(\cdot)$ is right continuous, it means that

$$\forall x \in \mathbb{R}, \quad \lim_{\varepsilon \downarrow 0} F_X(x+\varepsilon) = F_X(x)$$

epsilon is approaching zero

Let's consider that

$\varepsilon_n \geq 0$ is any non-negative sequence, such that

$\varepsilon \downarrow 0$ as $n \to \infty$

Now, consider the following

$$\lim_{n\to\infty} F_X(x+\varepsilon_n) = \lim_{n\to\infty} P(\{X \leq x+\varepsilon_n\})$$

we use ε_n here, as a trick so that we can deal with countable unions & intersections

As in, $\lim_{\varepsilon \downarrow 0} F_X(x+\varepsilon)$ we have uncountable ε as it is continuous and goes to zero

But, we cannot deal with uncountable unions or intersections

That is,

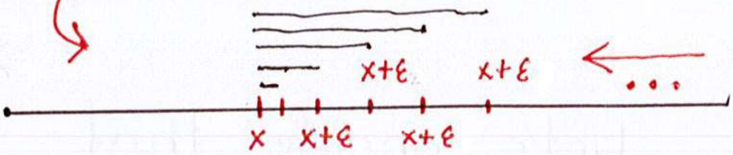
$$\lim_{\varepsilon \downarrow 0} F_{X}(x+\varepsilon) = \lim_{\varepsilon \downarrow 0} P(\{X \leq x+\varepsilon\})$$

$$= \lim_{n \to \infty} P(\{X \leq x+\varepsilon_n\})$$

We can write it in terms of ω as:

$$= \lim_{n \to \infty} P(\{\omega \mid X(\omega) \leq x+\varepsilon_n\})$$

$$= P\left(\bigcap_{n \in \mathbb{N}} \{\omega \mid X(\omega) \leq x+\varepsilon_n\}\right)$$



so, intersection keeps the smallest part from all the possibilities

$$= P(\{\omega \mid X(\omega) \leq x\}) = P(\{X \leq x\})$$

$$= F_{X}(x)$$

Generally, CDF need not be continuous, but the right-continuity must be satisfied. Any function that satisfies the right-continuity, monotonicity as in 2, the properties shown in 1, is eligible to be the CDF of some random variable.

Another CDF property:

Assume, $P(\{X = x_0\})$ stands for probability that X is equal to the particular number x_0. Then,

$$P(\{X = x_0\}) = F_X(x_0) - F_X(x_0^-)$$

where, • x_0^- is infinitesimally smaller point than x_0

• $F_X(x_0^-) = \lim_{\varepsilon \downarrow 0} F_X(x_0 - \varepsilon)$

We can prove it as follows:

$$
\begin{aligned}
& F_X(x_0) - F_X(x_0^-) \\
&= P_X((-\infty, x_0]) - \lim_{\varepsilon \to 0} F_X(x_0 - \varepsilon) \\
&= P_X((-\infty, x_0]) - \lim_{\varepsilon \to 0} P_X((-\infty, x_0 - \varepsilon]) \\
&= \lim_{\varepsilon \downarrow 0} P_X((x_0 - \varepsilon, x_0]) = P_X(\{X = x_0\})
\end{aligned}
$$

Definition: We recall the concept of random variable $\mathbb{X}$:

Given a random variable $\mathbb{X}$ defined on $(S, \mathcal{F}, P)$ that induces a new probability space $(\mathcal{R}, \mathcal{B}(\mathcal{R}), P_X)$, the cumulative distribution function (cdf) of $\mathbb{X}$ is defined as

$$
\begin{aligned}
F_{\mathbb{X}}(\alpha) &= P_{\mathbb{X}}((-\infty, \alpha]) = P_{\mathbb{X}}(\{x \mid x \leq \alpha\}) \\
&= P(\{\omega \in S \mid \mathbb{X}(\omega) \leq \alpha\}) \\
&= P(\mathbb{X}^{-1}((-\infty, \alpha])), \quad \alpha \in \mathbb{R} \\
&= P(\{\mathbb{X} \leq \alpha\}), \quad \alpha \in \mathbb{R}
\end{aligned}
$$

where, $\mathcal{R}$ is the range space of $\mathbb{X}$ and $\mathcal{R} \subset \mathbb{R}$
Aso, the cdf is the complete probabilistic description of $\mathbb{X}$.

As we already have seen that probabilistic description is a composition between $(\cdot)$ and $P(\cdot)$

$$P_{\mathbb{X}} = P \circ \mathbb{X}^{-1}$$

Properties of the cdf:

1. $F_{\mathbb{X}}(+\infty) = 1$ and $F_{\mathbb{X}}(-\infty) = 0$
2. If $x_1 < x_2$, then $F_{\mathbb{X}}(x_1) \leq F_{\mathbb{X}}(x_2)$
3. $P(\{\mathbb{X} > x\}) = 1 - F_{\mathbb{X}}(x), \quad \forall x \in \mathbb{R}$
4. If $x_1 < x_2$, then

$$P(\{x_1 < \mathbb{X} < x_2\}) = F_{\mathbb{X}}(x_2) - F_{\mathbb{X}}(x_1)$$

5. cdf is right-continuous. That is,

$F_{X}(\cdot)$ $\quad \forall x \in \mathbb{R} \quad \lim_{\varepsilon \downarrow 0} F_{X}(x+\varepsilon) = F_{X}(x)$

6. $P(\{X = x_0\}) = F_{X}(x_0) - F_{X}(x_0^-)$

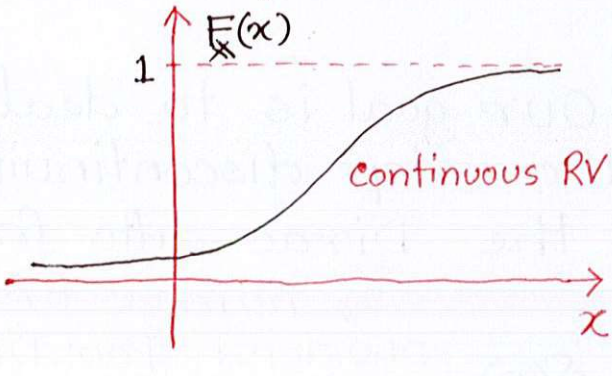
where, $\quad F_{X}(x_0^-) = \lim_{\varepsilon \downarrow 0} F_{X}(x_0 - \varepsilon)$

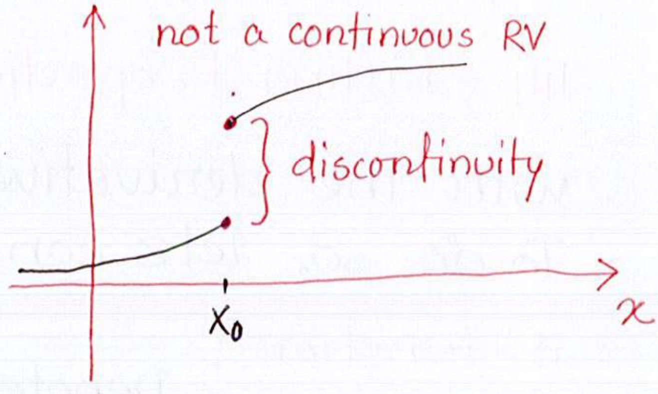
Types of Random Variables

Based on whether the CDF is continuous or not, random variables are subdivided into two subclasses:

1. Continuous random variable
2. Discrete random variable

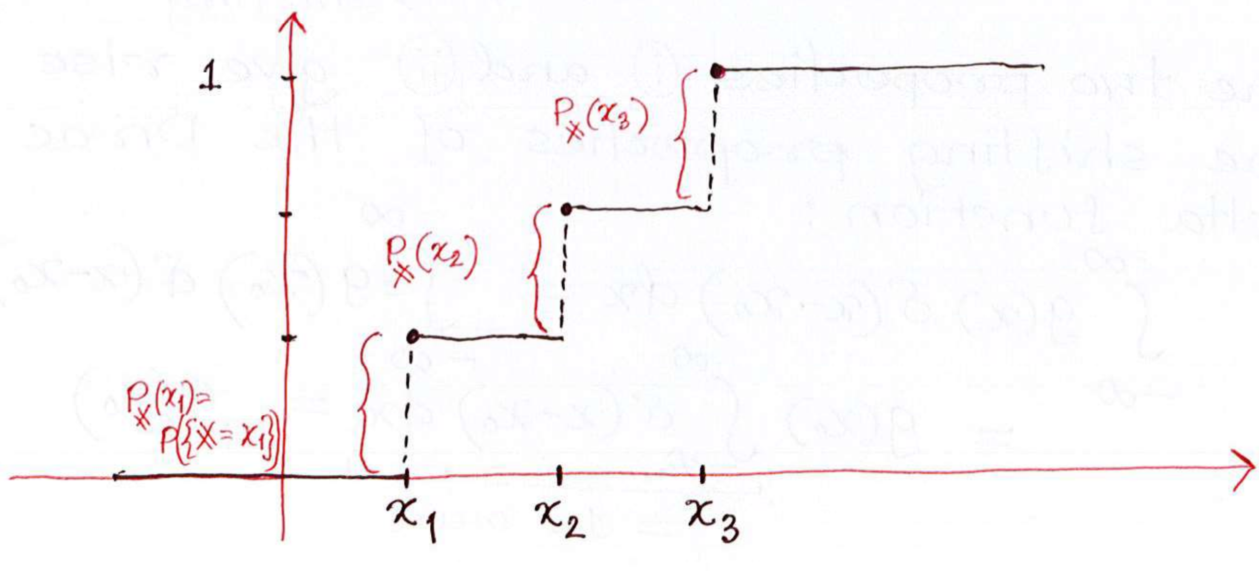
Definition: A random variable is absolutely continuous if $F_X(x)$, that is the cdf, is the continuous function of x. That is, continuous at all points $x \in \mathbb{R}$





Discrete RV: A random variable is discrete if the RV takes on values from a discrete (finite or countable) subset of $\mathbb{R}$.

For a discrete random variable the CDF is a staircase function.



Probability Density Function: The probability density function of a random variable is defined as the derivative of the cdf of the random variable with respect to (w.r.t) x:

$$f_X(x) = \frac{dF_X(x)}{dx}$$

where, $f_X(x) \geq 0 \quad \forall x \in \mathbb{R}$, and

$$\int_{-\infty}^{+\infty} f_X(x)\,dx = F_X(+\infty) - F_X(-\infty) = 1 - 0 = 1$$

Shifting Properties: Our goal is to deal with the derivative of the step discontinuities. To do so, let's consider the Dirac delta fn^c.
named on physicist Paul Dirac

δ-function: Denoted as $\delta(x)$ and defined as: (i) $\delta(x) = 0, \quad \forall x \neq 0$

(ii) $\displaystyle\int_{-\infty}^{\infty} \delta(x)\,dx = \int_{-\varepsilon}^{\varepsilon} \delta(x)\,dx = 1 \quad \forall \varepsilon > 0$

From (i) we can write

$$\underbrace{\delta(x - x_0)}_{\text{shifting}} = 0, \quad \forall x \neq x_0$$

The two properties (i) and (ii) give rise to the shifting properties of the Dirac delta function:

$$\int_{-\infty}^{\infty} g(x)\,\delta(x - x_0)\,dx = \int_{-\infty}^{\infty} g(x_0)\,\delta(x - x_0)\,dx$$

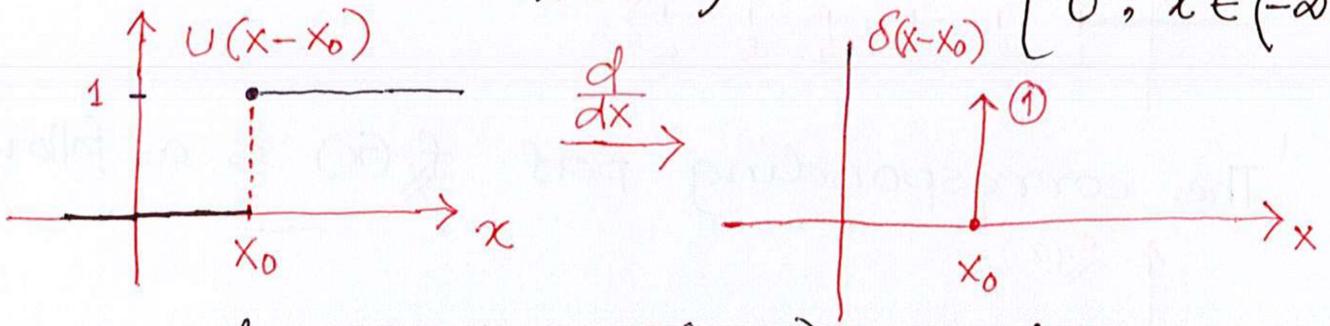
$$= g(x_0) \underbrace{\int_{-\infty}^{\infty} \delta(x - x_0)\,dx}_{= 1, \text{ area}} = g(x_0)$$

So, we obtain the shifting property as:

$$\int_{-\infty}^{\infty} g(x)\,\delta(x-x_0)\,dx = g(x_0)$$

Example: Suppose, we have

$$U(x-x_0) = 1_{[x_0,\infty)}(x) = \begin{cases} 1, & x \in [x_0,\infty) \\ 0, & x \in (-\infty, x_0) \end{cases}$$



So, $\frac{d}{dx} U(x-x_0) = \delta(x-x_0)$. That is:

$$V(x) = \int_{-\infty}^{x} \delta(r-x_0)\,dr = \begin{cases} 0, & x < x_0 \\ 1, & x > x_0 \\ 1, & x = x_0 \end{cases}$$

Consider a random variable that is the numerical outcomes of rolling a fair die. (each face is equally likely)

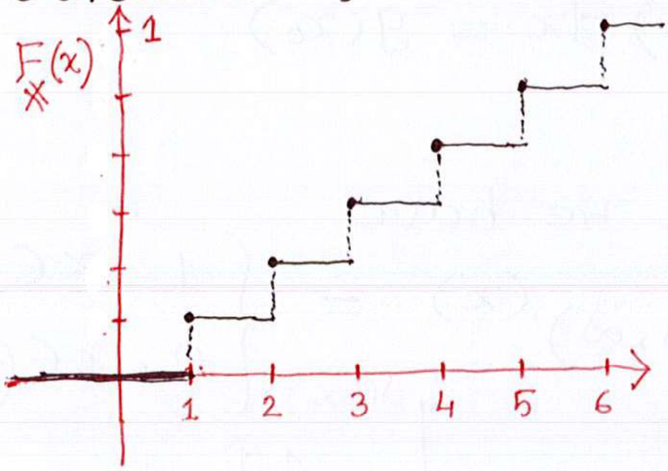
$$F_X(x) = P(\{X \leq x\})$$

$$= \frac{1}{6} 1_{[1,\infty)}(x) + \frac{1}{6} \cdot 1_{[2,\infty)}(x) + \frac{1}{6} \cdot 1_{[3,\infty)}(x) + \frac{1}{6} \cdot 1_{[4,\infty)}(x) + \frac{1}{6} \cdot 1_{[5,\infty)}(x) + \frac{1}{6} \cdot 1_{[6,\infty)}(x)$$

$$\Rightarrow f_X(x) = \frac{d}{d} F_X(x)$$

$$= \frac{1}{6}\delta(x-1) + \frac{1}{6}\delta(x-2) + \frac{1}{6}\delta(x-3) + \frac{1}{6}\delta(x-4) + \frac{1}{6}\delta(x-5) + \frac{1}{6}\delta(x-6)$$

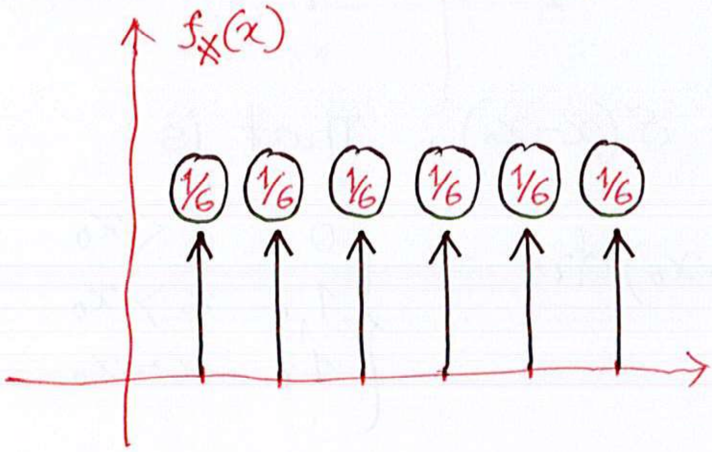
The plots of cdf and pdf of the numerical outcomes of the random variable



For instance,

$$P(\{\mathbb{X} < \quad\}) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

The corresponding pdf $f_{\mathbb{X}}(x)$ is as follows:



Properties of the pdf of a RV:

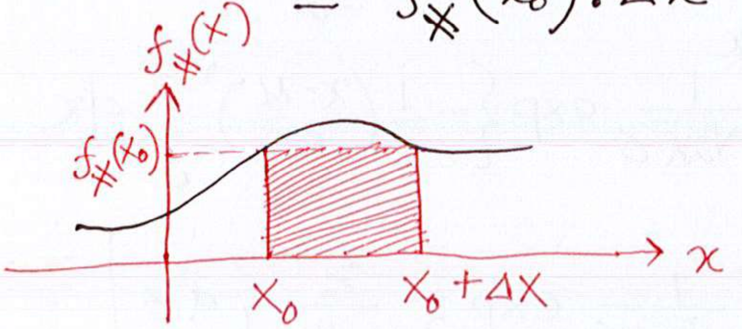
1. $f_{\mathbb{X}}(x) \geq 0, \quad \forall x \in \mathbb{R}$
2. $F_{\mathbb{X}}(x) = \int_{-\infty}^{x} f_{\mathbb{X}}(x)\, dx$
3. $\int_{-\infty}^{\infty} f_{\mathbb{X}}(x)\, dx = 1$
4. $P(\{x_1 \leq \quad \leq x_2\}) = \int_{x_1}^{x_2} f_{\mathbb{X}}(x)\, dx$

$$= F_{\mathbb{X}}(x_2) - F_{\mathbb{X}}(x_1)$$

For a continuous random variable $\mathbb{X}$,

$$P(\{x_0 \leq \mathbb{X} \leq x_0 + \Delta x\}) = \int_{x_0}^{x_0+\Delta x} f_{\mathbb{X}}(x)\,dx$$

$$\cong f_{\mathbb{X}}(x_0) \cdot \Delta x \quad \text{for small } \Delta x$$



As we see, accuracy increases as $\Delta x \to 0$

We can use the fundamental definition of derivative:

$$f_{\mathbb{X}}(x) = \frac{d}{dx} F_{\mathbb{X}}(x)$$

$$= \frac{F_{\mathbb{X}}(x+\Delta x) - F_{\mathbb{X}}(x)}{\Delta x}$$

$$\Rightarrow f_{\mathbb{X}}(x) \cdot \Delta x = F_{\mathbb{X}}(x+\Delta x) - F_{\mathbb{X}}(x)$$

$$= P(\{x < \mathbb{X} < x + \Delta x\})$$

Interestingly, we use pdf or cdf to describe RV and completely ignor the underlying $(S, \mathcal{F}, P)$

However, underlying $(S, \mathcal{F}, P)$ is there, we just don't think about the probality space and move on with the cdf or pdf.

Example: A random variable is known as a Gaussian random variable if it has a pdf of the form:

$$f_{\mathbb{X}}(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{-\frac{(x- \ )^2}{2\sigma^2}\right\}, \ \forall x \in \mathbb{R}$$

where, $\mu \in \mathbb{R}$ and $\sigma > 0$

So, using the pdf of Gaussian we can calculate the cdf as follows:

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx$$

$$= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\} dx$$

$$= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{z^2}{2}\right\} dz \qquad z = \frac{x-\mu}{\sigma}$$

$$= \Phi\left(\frac{x-\mu}{\sigma}\right)$$

$$\equiv G\left(\frac{x-\mu}{\sigma}\right)$$

Generally, $\Phi(\cdot)$ cannot be written in "closed form". It is numerically calculated and is widely tabulated.

So, if X is a Gaussian RV with parameters $\mu \in \mathbb{R}$ and $\sigma > 0$, then

$$P(\{a < X < b\}) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

Example: Uniformly distributed RV

A random variable X has a uniform distribution $X \sim U[a,b]$, $a<b$ if

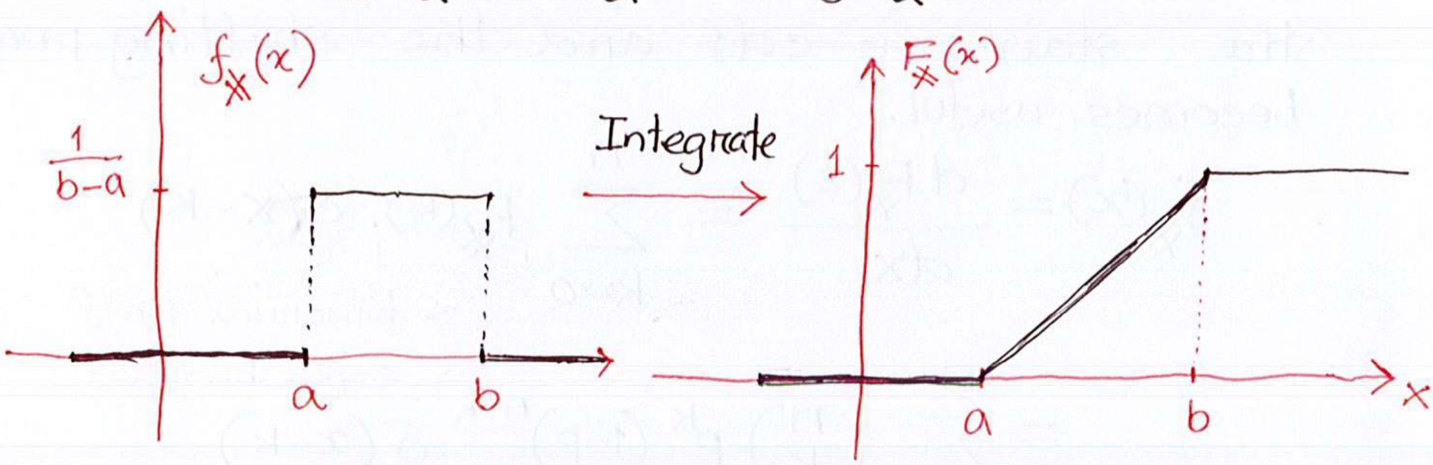
$$f_X(x) = \frac{1}{b-a} \cdot 1_{[a,b]}(x)$$

So, the cdf would be:

$$F_X(x) = \int_{-\infty}^{x} f_X(r) = \int_{-\infty}^{x} \frac{1}{b-a} \cdot 1_{[a,b]}(r)\, dr$$

$$= \frac{1}{b-a} \int_{-\infty}^{x} dr = \frac{1}{b-a} \int_{+a}^{x} dr$$

$$= \frac{1}{b-a} [r]_a^x = \frac{1}{b-a}(x-a)$$



Example: Binomially distributed RV

A binomially distributed RV is a discrete RV. It takes values from the set $\{0, 1, 2, \dots n\} \subset \mathbb{R}$ with pmf:

$$P_X(k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad \text{where}$$

$k = 0, 1, 2 \dots \dots n$

$p \in [0, 1]$

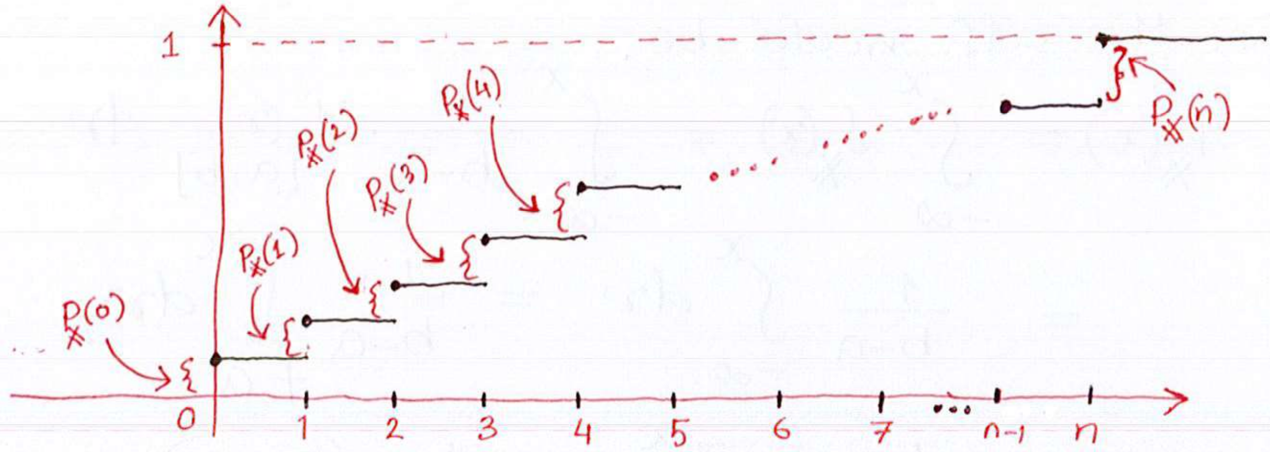
probability of occurrence of each favorable outcomes

The cdf of this RV can be calculated as:

$$F_X(x) = P(\{X \leq x\})$$

$$= \sum_{k=0}^{m(x)} \binom{n}{k} p^k (1-p)^{n-k}$$

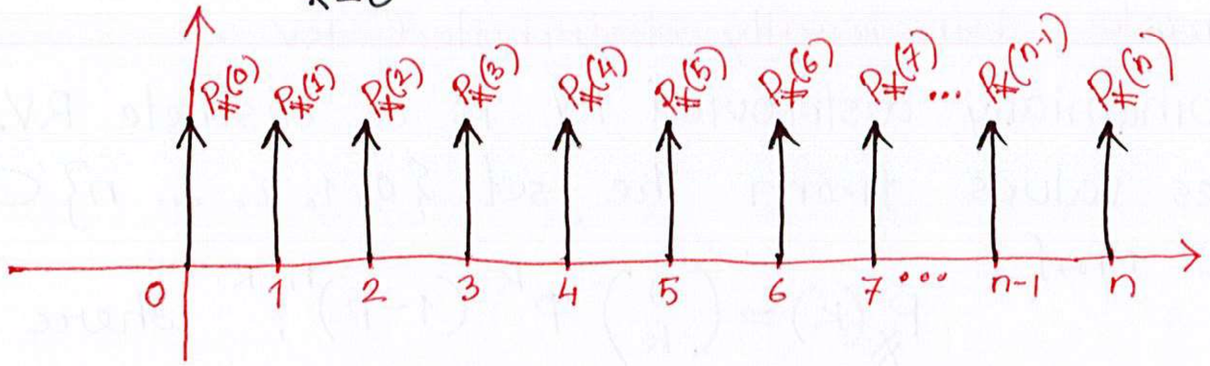
Here, $m(x)$ is an integer such that $m(x) \leq x < m(x)+1$



so, the pdf plot requires the derivative of the staircase cdf and the shifting property becomes useful.

$$f_X(x) = \frac{d F_X(x)}{dx} = \sum_{k=0}^{n} \underbrace{P_X(k)}_{\text{Binomial pmf could be}} \cdot \delta(x-k)$$

$$= \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \delta(x-k)$$

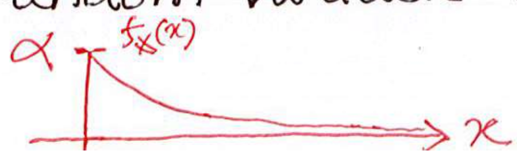


Example : Exponentially Distributed RV

A random variable with a pdf of the form

$$f_X(x) = \alpha e^{-\alpha x} \cdot 1_{[0,\infty)}(x) = \begin{cases} \alpha e^{-\alpha x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

where, $\alpha > 0$, is known as exponential random variable with parameter α.



Let's consider the case $M = \{X \leq a\}$, $a \in \mathbb{R}$

$$F_X(x/M) = P(\{X \leq x\} / \{X \leq a\})$$
$$= \frac{P(\{X \leq x\} \cap \{X \leq a\})}{P(\{X \leq a\})}$$

Now, when $x > a$,

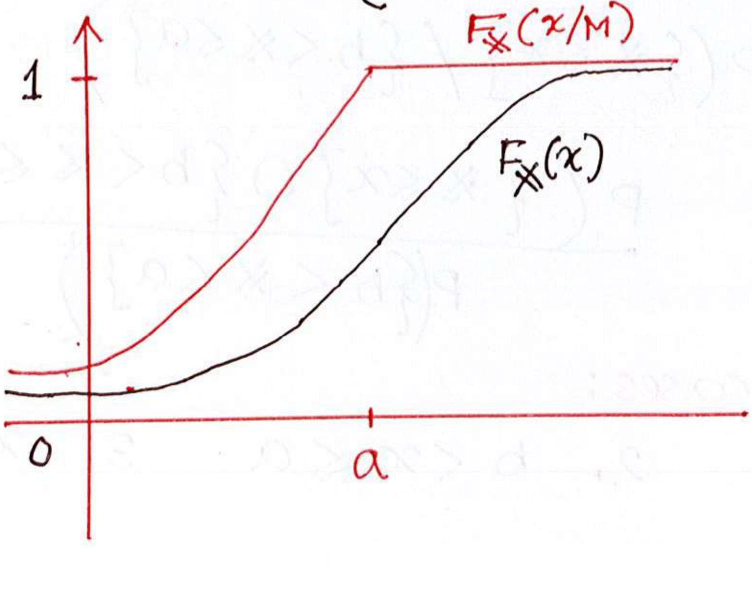
then $P(\{X \leq x\} \cap \{X \leq a\})$
$= P(\{X \leq a\})$

So, $$F_X(x/M) = \frac{P(\{X \leq a\})}{P(\{X \leq a\})} = \frac{F_X(a)}{F_X(a)} = 1$$

When $x \leq a$, then $P(\{X \leq x\} \cap \{X \leq a\})$
$= P(\{X \leq x\})$

So, $$F_X(x/M) = \frac{P(\{X \leq x\})}{P(\{X \leq a\})} = \frac{F_X(x)}{F_X(a)}$$

$$\therefore \quad F_X(x/M) = \begin{cases} F_X(x) & x \leq a \\ 1 & x > a \end{cases}$$



$F_X(a)$ is always smaller than or equal to 1, and hence, red line should be above the black line.

$\Downarrow$

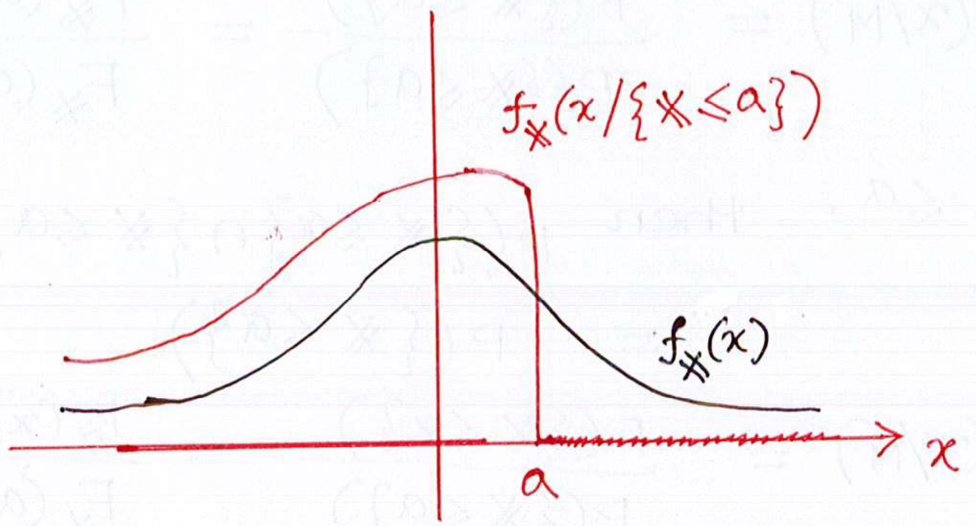
divided by less than one fact

The conditional pdf can be calculated by taking derivatives w.r.t x.

$$f_X(x|\{X \leq a\}) = \frac{dF_X(x|\{X \leq a\})}{dx}$$

$$= \begin{cases} \frac{f_X(x)}{F_X(a)}, & x \leq a \\ 0, & x > a \end{cases}$$

A pictorial representation goes as follows:



M coulbe defined as: $M = \{b < X \leq a\}$, $b<a$

so, $F_X(x/M) = F_X(x/\{b < X \leq a\})$

$$= P(\{X \leq x\} / \{b < X \leq a\})$$

$$= \frac{P(\{X \leq x\} \cap \{b < X \leq a\})}{P(\{b < X \leq a\})}$$

Three distinct cases:

1. $x > a$ 2. $b < x \leq a$ 3. $x \leq b$

Conditional Distributions

Given $(S, \mathcal{F}, P)$, assume that X is a random variable defined on the probability space $(S, \mathcal{F}, P)$. Consider that $A, M \in \mathcal{F}$, then

$$P(A/M) = \frac{P(A \cap M)}{P(M)}, \quad P(M) > 0$$

Let's assume, $A = \{X \leq x\} = \{\omega \in S \mid X(\omega) \leq x\}$

Then, $\underbrace{P(\{X \leq x\} \mid M)} = P(A/M)$

This is the conditional cdf of the RV X condition on event M

Definition: The conditional cdf of the RV X conditioned on $M \in \mathcal{F}$ is

$$F_X(x/M) \triangleq P(\{X \; x\}/M)$$
$$= \frac{P(\{X \leq x\} \cap M)}{P(M)}$$

The definition of $F_X(x/M)$ is similar to that of the $F_X(x)$, except for the fact that the conditional probability measure $P(\cdot/M)$ instead of the probability measure $P(\cdot)$. So,

$$\underset{\text{Valid probability measure}}{P(\cdot/M)} \rightarrow \underset{\text{is a valid cdf}}{F_X(x/M)}$$

$F_X(x/M)$ has all the properties of a valid cdf.

For instance $P(\{a < X \leq b\}/M) = F_X(b/M) - F_X(a/M)$

Definition: Conditional Probability Density fn^c

The conditional probability density fn^c of random variable X conditioned on $M \in \mathcal{F}$ is:

$$f_X(x/M) \triangleq \frac{d F_X(x/M)}{dx}$$

From the cdf $F_X(x/M)$, we can say that $f(x/M)$ is the valid pdf. That is,

$F_X(x/M)$ is a valid cdf $\longrightarrow$ $f_X(x/M)$ is a valid pdf

As $f_X(x/M)$ is a valid pdf, it also exhibit the necessary properties. For instance

$$P(\{a < X \leq b\}/M) = \int_a^b f_X(x/M)\, dx$$

Comment: In general, we must know the structure of $(S, \mathcal{F}, P)$ and the exact mapping being done through X to determine $F_X(x/M)$ o the conditional pdf $f_X(x/M)$.

Interestingly, Event M could be defined using the RV X. For instance,

1. $M = \{X \leq a\}$, $a \in \mathbb{R}$
2. $M = \{b < X \leq a\}$, $a, b \in \mathbb{R}$ and $b < a$

Conditional Distribution

Consider that we are interested in the probability of an event A, given that we already know M. Then, the definition of conditional probability gives —

$$P(A|M) \equiv \text{Probability of event A given that M has occurred}$$

$$= \frac{P(A \cap M)}{P(M)}, \quad P(M) > 0 \quad \cdots \text{①}$$

Let's consider further that the event $A = \{X \leq x\}$ That is, a random variable X has a value less than or equal to x. Equivalently,

$$A = \{X \leq x\} = \{w \in S \mid X(w) \leq x\}$$

Sample-space

outcome of the experiment

So, From Eq. 1, we obtain

$P(A/M) = P(\{X \leq x\} | M)$ which gives us the conditional Cumulative Distribution Function (CDF) for the random variable X defined on the event M. Here, $M \in \mathcal{F}$ and it has already occurred.

Definition: The conditional CDF of the Random Variable (RV) of X conditioned on the event $M \in \mathcal{F}$ is defined as

$$F_X(x|M) = P(\{X \leq x\} | M)$$

$$= \frac{P(\{X \leq x\} \cap M)}{P(M)}, \quad x \in \mathbb{R}, \; P(M) \neq 0 \text{ \& } > 0$$

The definition of the $F_X(x|M)$ is very similar to the definition of $F_X(x) \equiv P(\{X \leq x\})$, and the only difference is in the fact that we use conditional probability measure $P(\cdot|M)$ instead of the probability measure $P(\cdot)$.

In short. $P(\cdot/M) \Rightarrow F_X(\cdot/M)$

$P(\cdot/M)$ is a valid probability measure

$F_X(\cdot/M)$ is a valid cdf

so,

it must have all the properties of a valid cdf, defined earlier for $F_X(x)$.

Properties:

- $F_X(\infty/M) = 1$
- $F_X(-\infty/M) = 0$
- $$P(\{a < X \leq b\}/M) = \frac{P(\{a < X \leq b\} \cap M)}{P(M)}$$
$$= F_X(b/M) - F_X(a/M)$$

determine the term $F_X(x/M)$.

However, sometimes we can describe the event $M \in \mathcal{F}$ in terms of RV X. The two specific cases we consider here —

1. $M = \{X \leqslant a\}$, where $a \in \mathbb{R}$
2. $M = \{b < X \leqslant a\}$, $a, b \in \mathbb{R}$, $b < a$

Let's expand case 2 :

So, given that $M = \{b < X \leqslant a\}$ where $b < a$.

$$F_X(x/M) = P(\{X \leqslant x / M\})$$

$$= P(\{X \leqslant x / \{b < X \leqslant a\})$$

$$= \frac{P(\{X \leqslant x\} \cap \{b < X \leqslant a\})}{P(\{b < X \leqslant a\})} \quad \dots (1)$$

In Eq.ⁿ 1, we can have three cases —

1. $x > a$ 2. $b < x \leqslant a$, 3. $x \leqslant b$

When $x > a$:

$$F_X(x/M) = \frac{P(\{X \leqslant x\} \cap \{b < X \leqslant a\})}{P(\{b < X \leqslant a\})}$$

$$= \frac{P(\{X \leqslant a+\epsilon\} \cap \{b < X \leqslant a\}}{P(\{b < X \leqslant a\})}$$

For $X \leqslant a+\epsilon$

$$= \frac{P(\{b < X \leqslant a\})}{P(\{b < X \leqslant a\})}$$

$$= \frac{F_X(a) - F_X(b)}{F_X(a) - F_X(b)}$$

By definition of the CDF

$$\Rightarrow F_X(x/\{x>a\}) = 1$$

Conditional Probability Density Function

The conditional probability density function, by definition, is the derivative of the conditional CDF of a RV X conditioned on event M. Precisely,

$$f_X(x/M) \triangleq \frac{d}{dx} F_X(x/M)$$

Conditional PDF

Conditional CDF

By definition $F_X(x/M) = P(\{X \leq x\}/M)$. So,

$$\frac{d}{dx} F_X(x/M) = \lim_{\Delta x \to 0} \frac{P(\{x \leq X \leq x+\Delta x\}/M)}{\Delta x}$$

So, if we want to calculate the probability of an event $\{a \leq X \leq b\}$, conditional on M, then we write

$$P(\{a \leq X \leq b\}) = \int_a^b f_X(x/M)\, dx$$

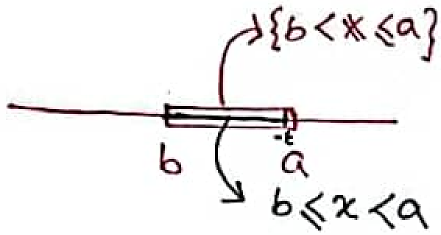
Comment: In general we must have the information on the probability space $(S, \mathcal{F}, P)$ and the exact mapping X to determine $F_X(x/M)$ and $f_X(x/M)$.

Now, to find $F(x/M)$ we must know the underlying experiment. However, if event M is expressed in terms of the random variable X, then knowing $F_X(x)$ is sufficient to ~~[illegible]~~

When $b \leq x < a$

$$F_X(x/M) = \frac{P(\{X \leq x\} \cap \{b < X \leq a\})}{P(\{b < X \leq a\})}$$

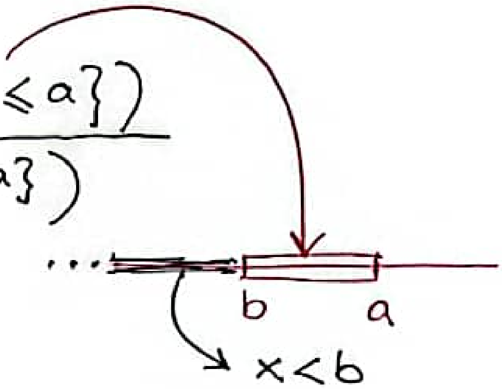
$$= \frac{P(\{b < x \leq x\})}{P(\{b\, X \leq a\})}$$



$$= \frac{F_X(x) - F_X(b)}{F_X(a) - F_X(b)} \quad \text{for} \quad b \leq x < a$$

When $x < b$

$$F_X(x/M) = \frac{P(\{X \leq x\} \cap \{b < X \leq a\})}{P(\{b < X \leq a\})}$$



$$= \frac{\phi}{P(\{b < X \leq a\})}$$

$$= \frac{\phi}{F_X(a) - F_X(b)} = 0 \quad \text{for} \quad x < b$$

Thus

$$F_X(x/M) = \begin{cases} 1 & \text{when } x > a \\ \dfrac{F_X(x) - F_X(b)}{F_X(a) - F_X(b)} & \text{when } b \leq x < a \\ 0 & \text{when } x < b \end{cases}$$

Taking derivative w.r.t x

$$f_X(x/M) = \frac{d}{dx} F_X(x/M) = \begin{cases} 0 & \text{when } x > a \\ 0 & \text{when } x < b \\ \dfrac{f_X(x)}{F_X(a) - F_X(b)} & \text{when } b < x \leq a \end{cases}$$