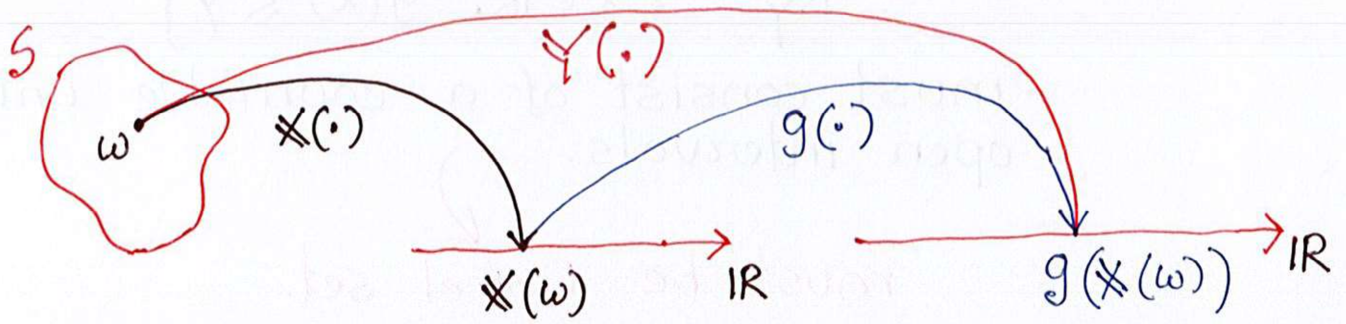
Functions of Random Variables

Let's assume that X is a random variable on the probability space $(S, \mathcal{F}, P)$.

We can use this random variable X and define a function Y as follows:

$$Y = g(X), \quad \text{where } g: \mathbb{R} \to \mathbb{R}$$



As we see, the mapping of ω is done to $\mathbb{R}$ twice when we define a function of random variable.

A composite function structure is evident from the definition

$$Y(\omega) = g(X(\cdot)), \text{ that is}$$

$$Y: S \to \mathbb{R}$$

However, the question is

Is $Y(\cdot)$ a random variable?

To assess it, let's recall the definition:

$Y: S \to \mathbb{R}$ is a random variable if

$$Y^{-1}(A) = \{\omega \in S \mid Y(\omega) \in A\} \in \mathcal{F},$$
$$\forall A \in \mathcal{B}(\mathbb{R})$$

where, $\mathcal{F}$ is the event space of $(S, \mathcal{F}, P)$

For $\mathbb{Y} = g(\mathbb{X})$ to be measurable (that is random variable) $g(\cdot)$ must satisfy the following properties

1. The domain of $g(\cdot)$ must contain the range space of $\mathbb{X}$
2. For each $y \in \mathbb{R}$, the set R_y defined as
$$R_y = \{x \in \mathbb{R};\ g(x) \leq y\}$$
must consist of a countable union of open intervals.

must be Borel set

3. The events $\{g(\mathbb{X}) = \pm\infty\}$ must have probability zero.

So, any function $g(\cdot)$ that satisfies these 3 properties is known as Baire function.

For such $g(\cdot)$, we can say that
$$\mathbb{Y} = g(\mathbb{X})$$
is a valid random variable.

Interestingly,

All functions we typically encounter in engineering applications are Baire functions.

Example: $Y = g(X) = X^2$

$\to fn^c$

The transformation considers $g(x) = x^2 = y$

As any specific y is the square of x, y will always be non-negative. So,

case $y < 0$: $F_Y(y) = 0, \quad y < 0$ [negative value of y is impossible]

For the case $y > 0$: Positive y

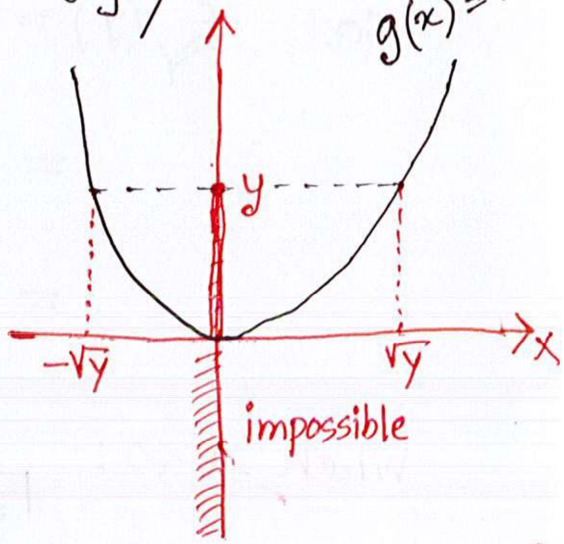
$$F_Y(y) = P(\{Y \le y\}) = P(\{X^2 < y\})$$

$$= P(\{-\sqrt{y} \le X \le +\sqrt{y}\})$$

$$= P(\{-\sqrt{y} < X \le +\sqrt{y}\})$$

or, cdf is continuous

for continuous RV case zero probability event can be written without the equal sign



$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

So, the CDF is

$$F_Y(y) = \left[F_X(\sqrt{y}) - F_X(-\sqrt{y})\right] \cdot 1_{(y, \infty)}(y)$$

What about $y = 0$? y becomes zero only at one point x, that is, $x = 0$. Now, if the cdf is continuous at zero, then the probability that $x = 0$ is equal to zero.

↳ So, we can ignore the case

↳ (y, ∞)

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

Examples: $\quad Y = g(X) = aX + b, \quad a, b \in \mathbb{R}$

Find $f_Y(y)$. Two cases $a>0$ and $a<0$

When $a>0$: $\quad F_Y(y) = P(\{Y \le y\}) = P(\{aX+b \le y\})$

$$= P\left(\left\{X \le \frac{y-b}{a}\right\}\right)$$

$$= F_X\left(\frac{y-b}{a}\right)$$

So, $\quad f_Y(y) = \frac{d}{dy} F_Y(y)$

$$= \frac{d}{dy} F_X\left(\frac{y-b}{a}\right) = f_X\left(\frac{y-b}{a}\right) \cdot \frac{d}{dy}\left(\frac{y-b}{a}\right)$$

$$= f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{a} = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

When $a<0$: $\quad F_Y(y) = P(\{Y \le y\})$

$$= P(\{aX+b \le y\}) = P\left(\left\{X \le \frac{y-b}{a}\right\}\right)$$

As $a<0$, it is negative, so,

$$= P\left(\left\{X \geq \frac{y-b}{a}\right\}\right)$$ because $a<0$, the inequality changes

$$= 1 - P\left(\left\{X \le \frac{y-b}{a}\right\}\right)$$

$$= 1 - F_X\left(\frac{y-b}{a}\right)$$

So, $\quad f_Y(y) = \frac{d}{dy}\left[1 - F_X\left(\frac{y-b}{a}\right)\right]$

$$\Rightarrow f_Y(y) = -f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{a} = -\frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

By combining $a<0$, $a>0$, we can write

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

So, $f_Y(y) = \dfrac{dF_Y(y)}{dy}$

$$\Rightarrow \quad f_Y(y) = \frac{d}{dy}\left[F_X(\sqrt{y}) - F_X(-\sqrt{y})\right] \cdot 1_{(y,\infty)}(y)$$

$$\Rightarrow \quad f_Y(y) = f_X(\sqrt{y})\frac{d}{dy}(\sqrt{y}) - f_X(-\sqrt{y})\frac{d}{dy}(-\sqrt{y})$$

$$= f_X(\sqrt{y})\frac{1}{2\sqrt{y}} - f_X(-\sqrt{y})\frac{-1}{2\sqrt{y}}$$

$$= \frac{1}{2\sqrt{y}}\left[f_X(\sqrt{y}) + f_X(-\sqrt{y})\right]$$

$$= \frac{1}{2\sqrt{y}}\left[f_X(\sqrt{y}) + f_X(-\sqrt{y})\right] \cdot 1_{(0,\infty)}(y)$$

Function of random variables

The direct pdf method:

Suppose $Y = g(X)$, where $g: \mathbb{R} \to \mathbb{R}$ such that $g^{-1}(\cdot)$ exists. That is $y = g(x)$

$\Rightarrow x = g^{-1}(y)$

$\Rightarrow x(y) = g^{-1}(y)$

It is also assumed that

$$\frac{dx}{dy} = \frac{d\,g^{-1}(y)}{dy} \text{ exists.}$$

Then, $$f_Y(y) = f_X(g^{-1}(y)) \cdot \left| \frac{d\,g^{-1}(y)}{dy} \right|$$

where, $x(y) = g^{-1}(y)$

Example: Given that $X \sim U[0,1]$ and let's assume that $Y = g(X) = \sqrt{X}$. So, find $f_Y(y)$.

Let's apply direct pdf method

According to the formula: $$f_Y(y) = f_X(x(y)) \left| \frac{dx(y)}{dy} \right| = f_X(g^{-1}(y)) \left| \frac{d\,g^{-1}(y)}{dy} \right|$$

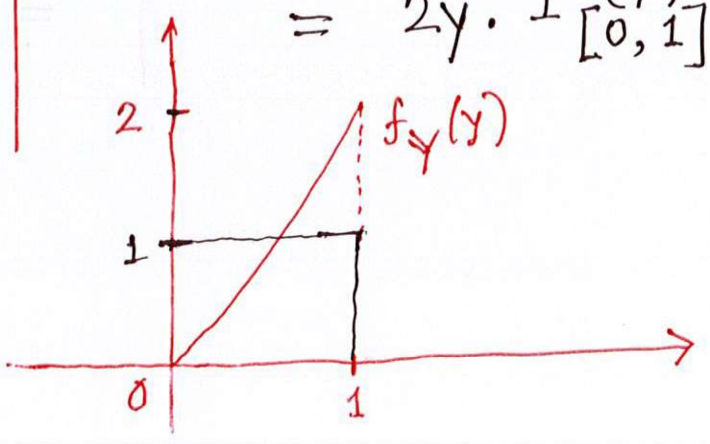
Here, $y = g(x) = \sqrt{x}$

$\Rightarrow x = y^2$

$\Rightarrow x(y) = y^2$

So, $$\frac{dx(y)}{dy} = 2y$$

So, $$f_Y(y) = |2y| \cdot f(x(y)) = 2y \cdot 1_{[0,1]}(y)$$



□ Example: Consider X be a Gaussian Random variable with pdf $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

mean μ : 0
variance σ^2 : 1

$N(\mu, \sigma^2)$ is the standard notation for the Gaussian Random Variable with mean μ and variance σ^2.

Consider the linear transformation $Y = aX + b$
Let's find out $f_Y(y)$ using the direct pdf method.

$$f_Y(y) = f_X(x(y)) \left| \frac{dx(y)}{dy} \right|$$

$y = ax + b$

$\Rightarrow x = \frac{y-b}{a}$

$\Rightarrow \underbrace{x(y)}_{\text{rewriting con dering } x \text{ as a fn}^c \text{ of } y} = \frac{y-b}{a}$

$$\frac{dx(y)}{dy} = \frac{d}{dy}\left(\frac{y-b}{a}\right) = \frac{1}{a}$$

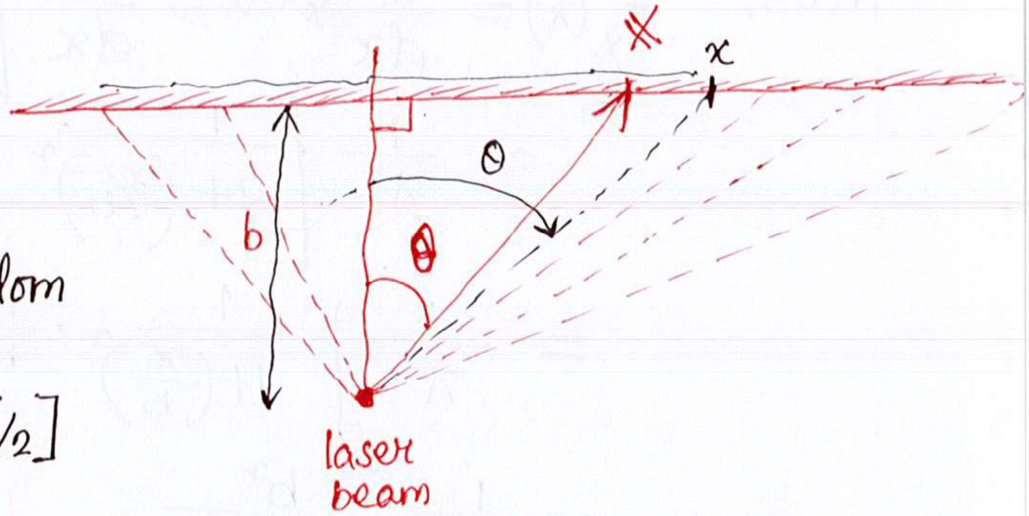
$$\therefore f = f_X(x(y)) \left| \frac{1}{a} \right|$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-(y-b)^2}{2a^2}\right) \cdot \frac{1}{|a|}$$

$$= \frac{1}{\sqrt{2\pi}\,|a|} \exp\left\{\frac{-(y-b)}{2a^2}\right\}$$

Example: Suppose that a laser beam points at infinitely long wall, and the wall is at a distance b.

Assume that θ is uniformly distributed random variable on the interval $[-\pi/2, \pi/2]$



Let's say X is the coordinate along the wall where the laser hits. Find the pdf of X

Solution: Let's find out $F_X(x)$ first.

$$\tan\theta = \frac{x}{b}$$
$$\Rightarrow \theta = \tan^{-1}\frac{x}{b}$$
$$\Rightarrow x = b\tan\theta$$
$$\Rightarrow x(\theta) = b\tan\theta$$

random, changes for θ random

So, $P(X \leq x)$
$$= P(b\tan\theta \leq x)$$
$$= P\left(\theta \leq \tan^{-1}\frac{x}{b}\right)$$
$$= F_\theta(\tan^{-1} x/b)$$

Now, $F_X(x) = P\{X \leq x\}$
$$= F_\theta(\tan^{-1} x/b)$$
$$= \frac{\theta \text{ values generating wall coordinate less than or equal to } x}{\text{total possible } \theta \text{ values}}$$
$$= \frac{\tan^{-1}(x/b) + \pi/2}{\pi}$$

So, $F_X(x) = \frac{1}{\pi}\left[\tan^{-1}(x/b) + \frac{\pi}{2}\right]$

Thus, $f_X(x) = \frac{d F_X(x)}{dx} = \frac{d}{dx}\left[\frac{1}{\pi}\left(\tan^{-1} x/b + \frac{\pi}{2}\right)\right]$

$$= \frac{1}{\pi}\left[\frac{1}{1+(x/b)^2} \cdot \frac{d}{dx}(x/b) + 0\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{1+\left(\frac{x}{b}\right)^2} \cdot \frac{1}{b}\right] = \frac{1}{\pi b}\left(\frac{1}{\frac{b^2+x^2}{b^2}}\right)$$

$$= \frac{1}{\pi b} \frac{b^2}{b^2+x^2}$$ **Answer**

Question : Assume that X is a random variable uniformly distributed on $(0, 1)$ and let $Y = g(X)$, where

$$g(x) = -\lambda \ln(1-x)$$

and $\lambda > 0$

Here, $f_X(x) = 1_{[0,1]}^{(x)}$

According to direct method : $f_Y(y) = f_X(x(y)) \left|\frac{dx(y)}{dy}\right|$

So, $y = g(x) = -\lambda \ln(1-x)$

$\Rightarrow -\frac{y}{\lambda} = \ln(1-x)$

$\Rightarrow 1-x = e^{-y/\lambda}$

$\Rightarrow x = 1-e^{-y/\lambda}$

$\Rightarrow x(y) = 1-e^{-y/\lambda}$

$\frac{d}{dy} x(y) = \frac{d}{dy}(1-e^{-y/\lambda})$

$= -e^{-y/\lambda} \cdot \lambda)(-1)$

$= e^{-y/\lambda} \cdot 1/\lambda$

So, $f_Y(y) = e^{-y/\lambda} \cdot \frac{1}{\lambda} \cdot 1_{[0,1]}^{(y)}$