Reading Assignment: Sections 1-1, 1-2, 1-3, 2-1, and 2-2. 2-3 of Papoulis Text Book

Q1: Reduce the following expressions to the simplest possible forms:

(a) $(A \cap \overline{B}) \cup (B \cap \overline{A})$.

(b) $(A \cap \overline{B}) \cap (A \cap B)$.

(c) Use DeMorgan's laws to show that:

  (a) $\overline{A \cap (B \cup C)} = (\overline{A} \cup \overline{B}) \cap (\overline{A} \cup \overline{C})$.

  (b) $\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$.

Q2: Let $\{A_1, \ldots, A_n\}$ be a partition of the space $\mathcal{S}$, and define the family of sets $\{B_1, \ldots, B_n\}$ by $B_j = G \cap A_j, \quad j = 1, \ldots, n$, where $G \subset \mathcal{S}$. Show that $\{B_1, \ldots, B_n\}$ is a partition of the set G.

Q3: Prove that a finite set with n elements has 2^n distinct subsets.

Q4: Using the definitions of union, intersection, and complement, show that

(a) $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

(b) $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Q5: (*Papoulis,* Problem 2-2) If $A = \{2 \leq x \leq 5\}$ and $B = \{3 \leq x \leq 6\}$, Find $A \cup B$, $A \cap B$, and $(A \cup B) \cap (\overline{A \cap B})$.

Q6: (*Papoulis,* Problem 2-3) Show that if $A \cap B = \emptyset$, then $P(A) \leq P(\overline{B})$.

Q7: (*Papoulis,* Problem 2-4) Show that

(a) if $P(A) = P(B) = P(A \cap B)$, then $P((A \cap \overline{B}) \cup (B \cap \overline{A})) = 0$;

(b) if $P(A) = P(B) = 1$, then $P(A \cap B) = 1$.

Q8: (*Papoulis,* Problem 2-5.) Prove and generalize the following identity: $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$. By generalize, we mean to the union of n events.

Q9: (*Papoulis,* Problem 2-6) Show that if sample space $\mathcal{S}$ of a random experiment consists of a countable number of outcomes ξ_i and each subset $\{\xi_i\}$ is an event in the event space, then every subset of $\mathcal{S}$ is an event in the event space of the random experiment.

Q10: (*Papoulis,* Problem 2-7) If $\mathcal{S} = \{1, 2, 3, 4\}$ is the sample space of a random experiment, find the smallest σ-field that contains the events $\{1\}$ and $\{2, 3\}$.

Q11: (*Papoulis,* Problem 2-8) If $A \subset B$, P(A)=1/4, and $P(B) = 1/3$, find $P(A|B)$ and $P(B|A)$.

Q12: (*Papoulis,* Problem 2-9) Show that $P(A \cap B|C) = P(A|B \cap C)P(B|C)$ and $P(A \cap B \cap C) = P(A|B \cap C)P(B|C)P(C)$.

Q13: Show that for any two events A and B in a probability space $(\mathcal{S}, \mathcal{F}, P)$ the following relationship holds: $P(A)P(B) - P(A \cap B) = P(\overline{A} \cap B) - P(\overline{A})P(B) = P(A \cap \overline{B}) - P(A)P(\overline{B})$.

Q14: Express each of the below events in terms of the events A, B, and C and the operations of complementation, union, and intersection:

(a) at least one of the events A, B, C occurs;

(b) at most one of the events A, B, C occurs;

(c) none of the events A, B, C occurs;

(d) all three events occur;

(e) exactly one of the events A, B, C occurs;

(f) A and B occur, but not C;

(g) A occurs, if not then B does not occur either.

Q15: Let $\mathcal{S}$ be the sample space corresponding to the random experiment of tossing a coin three times and noting the sequence of H and T (*heads* and *tails*). Let A be the event that head occurs exactly twice, let B be the event that at least two heads appear, and let C be the event that head appears when tail has appeared at least once.

(a) Give the elements of A, B, and C;

(b) Describe the events: (i) $\overline{A} \cap B$, (ii) $\overline{A} \cap \overline{B}$, (iii) $A \cap C$.