Mean, Variance and Expectation

Definition: The mean or expected value of a random variable X with pdf $f_X(x)$ is defined as:

$$E[X] \triangleq \int_{-\infty}^{\infty} x\, f_X(x)\, dx \quad \cdots \cdots (1)$$

While the definition as in (1) is for continuous RV, the definition can be extended for discrete RV as well.

For instance, pdf as δ-function

If $P(\{X = x_K\}) = P_X(x_K) = P_K$ over a discrete index set, then

$$f_X(x) = \sum_K P_X(x_K)\, \delta(x - x_K) = \sum_K P_K\, \delta(x - x_K)$$

and,

$$E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\, dx = \int_{-\infty}^{\infty} x \left(\sum_K P_K\, \delta(x - x_K) \right) d$$

$$= \sum_K P_K \int_{-\infty}^{\infty} x\, \delta(x - x_K)\, dx = \sum_K P_K x_K$$

$$= \sum_K P_X(x_K)\, x_K$$ Mean for discrete RV

So, for a discrete RV X, we have

$$E[X] = \sum_K x_K\, P_X(x_K)$$

Conditional mean of RV

Let's consider $\mathbb{X}$ be a RV on the probability space $(S, \mathcal{F}, P)$ and M is an event in the event space.

Then, the conditional mean of $\mathbb{X}$ conditioned on M is

$$E[\mathbb{X}|M] \triangleq \int_{-\infty}^{\infty} x f_{\mathbb{X}}(x/M)\, dx$$

However, if $\mathbb{X}$ is a discrete RV, the conditional pmf $P_{\mathbb{X}}(x_K/M) = P(\{\mathbb{X}=x_K\}/M)$, then

$$E[\mathbb{X}/M] = \int_{-\infty}^{\infty} x f_{\mathbb{X}}(x/M)\, dx = \int_{-\infty}^{\infty} x\left(\sum_K P_{\mathbb{X}}(x_K/M)\,\delta(x-x_K)\right) dx$$

$$= \sum_K P_{\mathbb{X}}(x_K/M) \cdot \underbrace{\int_{-\infty}^{\infty} x\,\delta(x-x_K)\, dx}_{\text{dirac-}\delta\text{ property}}$$

$$= \sum_K x_K P_{\mathbb{X}}(x_K/M)$$

◫ Mean : exponential

Let X be an exponentially distributed RV with pdf $f_X(x) = \frac{1}{\mu} \exp\left\{-\frac{x}{\mu}\right\} 1_{[0,\infty]}(x)$, $\mu > 0$

What is $E[X]$?

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_0^{\infty} \underbrace{x}_{u} \cdot \frac{1}{\mu} \underbrace{e^{-x/\mu}}_{v} dx$$

$$= \int uv\, dx,$$ where, u is a fn^c of x, v is a fn^c of x

$$= \frac{1}{\mu}\left[x \int e^{-x/\mu} dx - \int \frac{d}{dx} x \left(\int e^{-x/\mu} dx\right) dx\right]_0^{\infty}$$

$$= \frac{1}{\mu}\left[-x \cdot e^{-x/\mu} \cdot \mu - \int -\mu e^{-x/\mu} dx\right]_0^{\infty}$$

$$= \left[-x e^{-x/\mu} + \int e^{-x/\mu}\right]_0^{\infty}$$

$$= \left[-x e^{-x/\mu} - \mu e^{-x/\mu}\right]_0^{\infty}$$

$$= -\left(-\mu e^{-0/\mu}\right) = \mu e^{-0} = \mu$$ Mean

We can use the same X and conditioned on M.

What would be the expression for $E[X/M]$, where $M = \{X > \mu\}$

According to expectation formula:

$$E[X/M] = E[X/\{X > \mu\}]$$

$$= \int_{-\infty}^{\infty} x f_X(x/\{X > \mu\})\, dx$$

We can calculate $f_{\mathbb{X}}(x / \underbrace{\{\mathbb{X} > \mu\}}_{M})$ by evaluating the cdf $F_{\mathbb{X}}(x/M)$.

Then, we can take derivative of $F_{\mathbb{X}}(x/M)$ to obtain the cdf.

We can use the formula as follows:

$$F_{\mathbb{X}}(x/M) = P(\{\mathbb{X} \leq x\}/M)$$

$$= \frac{P(\{\mathbb{X} \leq x\} \cap M)}{P(M)} = \frac{P(A/\{\mathbb{X} \leq x\}) F_{\mathbb{X}}(x)}{P(\;)}$$

If, $M = \{\mathbb{X} > \mu\}$

$$= \frac{P(\{\mathbb{X} > \mu\}/\{\mathbb{X} \leq x\}) F_{\mathbb{X}}(x)}{P(\{\mathbb{X} > \mu\})}$$

Question is, How can you calculate $F_{\mathbb{X}}(x)$ from the given $f_{\mathbb{X}}(x)$?

we need this for $F_{\mathbb{X}}(x)$ calculation

$$F_{\mathbb{X}}(x) = \int f_{\mathbb{X}}(x)\, dx$$

$$F_{\mathbb{X}}(x/M) = \int f_{\mathbb{X}}(x/M)\, dx$$

$$P(\{a < \mathbb{X} \leq b\}/M) = F_{\mathbb{X}}(b/M) - F_{\mathbb{X}}(a/M)$$

We can write:

$$P(\{a < \mathbb{X} \leq b\}/M) = \int_a^b f_{\mathbb{X}}(x/M)\, dx$$

So, $E[\mathbb{X}/\{\mathbb{X}>\mu\}] = \mu + \mu = 2\mu$. As we obtain $E[\mathbb{X}] \neq E[\mathbb{X}|\{\mathbb{X}>\mu\}]$. Generally,

$$E[\mathbb{X}] \neq E[\mathbb{X}/M], \quad M \in \mathcal{F}$$

RV as a fn^c of another RV : Expectation

Suppose we have a RV $\mathbb{X}$ defined on $(S, \mathcal{F}, P)$ and the pdf is $f_{\mathbb{X}}(x)$.

Consider that a new random variable is defined as: $\mathbb{Y} = g(\mathbb{X})$, where $g: \mathbb{R} \to \mathbb{R}$

So, calculate $E[\mathbb{Y}] = E[g(\mathbb{X})]$, where $g: \mathbb{R} \to \mathbb{R}$

According to the formula, we can calculate as

$$E[\mathbb{Y}] = \int_{-\infty}^{\infty} y \underbrace{f_{\mathbb{Y}}(y)}_{\text{we need evaluation of this.}} dy$$

we can use the below result to calculate $E[\mathbb{Y}]$. So,

$$E[g(\mathbb{X})] \triangleq \int_{-\infty}^{\infty} g(x) f_{\mathbb{X}}(x)\, dx$$

For a discrete random variable, $f_{\mathbb{X}}(x) = \sum_k P_{\mathbb{X}}(x_k)\,\delta(x-x_k)$

$$E[g(\mathbb{X})] = \int_{-\infty}^{\infty} g(x) \left(\sum_k P_{\mathbb{X}}(x_k)\,\delta(x-x_k) \right) dx$$

$$= \sum_k P_{\mathbb{X}}(x_k) \int_{-\infty}^{\infty} g(x)\,\delta(x-x_k)\,dx = \sum_k g(x_k) P_{\mathbb{X}}(x_k)$$

So, let's calculate the conditional mean $E[X/M]$ for a random variable with pdf

$$f_X(x) = \frac{1}{\mu} \exp\left\{-\frac{x}{\mu}\right\} \cdot 1_{[0,\infty)}(x)$$

$$E[X/M] = \int x f_X(x/M)\, dx$$

$$= \int x\, f_X(x/\{X>\mu\})\, dx$$

For $M = \{X > \mu\}$

We can show that

$$f_X(x/\{X>\mu\}) = \frac{\frac{1}{\mu}\exp\left\{-\frac{x}{\mu}\right\} \cdot 1_{(\mu,\infty)}(x)}{e^{-\mu/\mu}}$$

$$= \frac{1}{\mu} \exp\left\{-\frac{(x-\mu)}{\mu}\right\} 1_{(\mu,\infty)}(x)$$

$$\therefore\ E[X|\{X>\mu\}] = \int_{-\infty}^{\infty} x\, f_X(x|\{X>\mu\})\, dx$$

$$= \int_{\mu}^{\infty} x \cdot \frac{1}{\mu} \exp\left\{\frac{-(x-\mu)}{\mu}\right\} dx$$

Consider $r = x - \mu$

$\Rightarrow x = r + \mu$

$\Rightarrow dx = dr$

So, we obtain,

$$= \int_0^{\infty} (r+\mu) \frac{1}{\mu} \exp\left\{\frac{-r}{\mu}\right\} dr$$

$$= \int_0^{\infty} r \cdot \frac{1}{\mu} e^{-\left\{\frac{r}{\mu}\right\}} dr + \int_0^{\infty} \mu \cdot \frac{1}{\mu} e^{-\left\{\frac{r}{\mu}\right\}} \cdot dr$$

$$= \underbrace{\frac{1}{\mu}\int_0^{\infty} r \cdot e^{-\left\{\frac{r}{\mu}\right\}} dr}_{\mu} + \int_0^{\infty} e^{-\frac{r}{\mu}} dr$$

From $E[X]$ derivation

$$\left[-\mu e^{-x/\mu}\right]_0^{\infty} = +\mu \cdot e^0 = \mu \cdot 1 = \mu$$

Variance: Given a random variable $\mathbb{X}$, the variance of $\mathbb{X}$, denoted as $Var(\mathbb{X})$, is as follows

$$Var(\mathbb{X}) \triangleq E\left[(\mathbb{X}-\bar{\mathbb{X}})^2\right]$$
$$= \int_{-\infty}^{\infty} (x-\bar{\mathbb{X}})^2 f_{\mathbb{X}}(x)\,dx$$

Here, $\bar{\mathbb{X}} = E[\mathbb{X}]$

often $\sigma_{\mathbb{X}}^2$ is used as the symbol for $Var(\mathbb{X})$

$$\sigma_{\mathbb{X}}^2 = Var(\mathbb{X})$$

Positive square root of the variance of $\mathbb{X}$ is called the standard deviation of $\mathbb{X}$.

$$\text{Std. Dev}(\mathbb{X}) = \sqrt{\sigma_{\mathbb{X}}^2} = \sigma_{\mathbb{X}} = \sqrt{Var(\mathbb{X})}$$

$$Var(\mathbb{X}) = E\left[(\mathbb{X}-\bar{\mathbb{X}})^2\right] = E\left[\mathbb{X}^2 - 2\mathbb{X}\bar{\mathbb{X}} + \bar{\mathbb{X}}^2\right]$$
$$= E[\mathbb{X}^2] - E[2\mathbb{X}\bar{\mathbb{X}}] + E[\bar{\mathbb{X}}^2]$$
$$= E[\mathbb{X}^2] - 2\bar{\mathbb{X}}\,E[\mathbb{X}] + E[\bar{\mathbb{X}}^2]$$
$$= E[\mathbb{X}^2] - 2\bar{\mathbb{X}}\cdot\bar{\mathbb{X}} + \bar{\mathbb{X}}^2$$
$$= E[\mathbb{X}^2] - \bar{\mathbb{X}}^2 = E[\mathbb{X}^2] - \left(E[\mathbb{X}]\right)^2$$

So, $Var(\mathbb{X}) = E[\mathbb{X}^2] - \left(E[\mathbb{X}]\right)^2$

Linearity of expectation

Consider that $g_1(X)$ and $g_2(X)$ are the functions of RV X, and α, β are the constants with $\alpha, \beta \in \mathbb{R}$

Then, $$E[\alpha g_1(X) + \beta g_2(X)] = \alpha E[g_1(X)] + g_2(X)\beta$$

Example: Consider a Gaussian RV X with pdf

$\mu \in \mathbb{R}$ and $\sigma > 0$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$$

Find $E[X]$ and $Var(X)$

$$E[X] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} \cdot x \, dx$$

Let $x - \mu = r$
$\Rightarrow x = \mu + r$
$\Rightarrow dx = d\mu + dr$
$\Rightarrow dx = dr$

$$= \int_{-\infty}^{\infty} (\mu + r) \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dr$$

$$= \int_{-\infty}^{\infty} (\mu + r) \exp\left\{-\frac{r^2}{2\sigma^2}\right\} dr$$

$$= \underbrace{\int_{-\infty}^{\infty} r \exp\left\{-\frac{r^2}{2\sigma^2}\right\} d}_{\text{expand it Part A}} + \int_{-\infty}^{\infty} \mu \exp\left\{-\frac{r^2}{2\sigma^2}\right\} dr$$

Part A: $$\frac{1}{\sqrt{2\pi}\sigma}\left[\int_{-\infty}^{0} r \exp\left\{-\frac{r^2}{2\sigma^2}\right\} dr + \int_{0}^{\infty} r \exp\left\{-\frac{r^2}{2\sigma^2}\right\} dr\right]$$

$$= \frac{1}{\sqrt{2\pi}\sigma}\left[-\int_{0}^{-\infty} r \exp\left\{-\frac{r^2}{2\sigma^2}\right\} dr + \int_{0}^{\infty} r \exp\left\{-\frac{r^2}{2\sigma^2}\right\} dr\right]$$

We can rewrite by doing a change of variable.

So, $\frac{1}{\sqrt{2\pi}\sigma}\int_0^{-\infty}(-r)\exp\left\{-\frac{r^2}{2\sigma^2}\right\}dr$

$$=\frac{1}{\sqrt{2\pi}\sigma}\int_0^{-\infty}(-r)\exp\left\{-\frac{(-r)^2}{2\sigma^2}\right\}dr$$

$$=\frac{1}{\sqrt{2\pi}\sigma}\int_0^{\infty} r\exp\left\{-\frac{r^2}{2\sigma^2}\right\}(-dr)$$

consider $(-r) = r$

so, $dr = -dr$

When $-r = 0$
$\Rightarrow r = 0$
$-r = -\infty$
$\Rightarrow r = \infty$

after the change of variable process

limits are changed

$$=\frac{1}{\sqrt{2\pi}\sigma}\cdot(-1)\cdot\int_0^{\infty} r\exp\left\{-\frac{r^2}{2\sigma^2}\right\}dr$$

$$=-\frac{1}{\sqrt{2\pi}\sigma}\int_0^{\infty} r\exp\left\{-\frac{r^2}{2\sigma^2}\right\}dr$$

By plugging in these values, we obtain

$$\text{Part A} = \frac{1}{\sqrt{2\pi}\sigma}\left[\int_0^{\infty}-r\exp\left\{-\frac{r^2}{2\sigma^2}\right\}dr+\int_0^{\infty} r\exp\left\{-\frac{r^2}{2\sigma^2}\right\}dr\right]$$

$$= 0$$

That is,

$$E[X] = \int_{-\infty}^{\infty}\mu\cdot\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{r^2}{2\sigma^2}\right\}dr$$

We can do change of variable by assuming

$$\frac{r^2}{2\sigma^2} = z$$

So, $z = \frac{x}{\sqrt{2}\sigma}$ | $\Rightarrow x^2 = 2z\sigma^2$

$\Rightarrow dz = \frac{1}{\sqrt{2}\sigma} dx$

$\Rightarrow dz.\sqrt{2}\sigma = dx$ | $\Rightarrow \sqrt{2}\sigma\, dz = dx$

Replacing the variable x-term by z, we obtain

$$E[X] = \int_{-\infty}^{\infty} \mu . \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\left(\frac{x}{\sqrt{2}\sigma}\right)^2\right\} dx$$

$$= \int_{-\infty}^{\infty} \mu . \frac{1}{\sqrt{2\pi}\sigma} e^{-z^2} . dz.$$

$$= \int_{-\infty}^{\infty} \mu . \frac{1}{\sqrt{\pi}} e^{-z^2} dz$$

$$= \mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-z^2} dz = \mu . \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-z^2} dz$$

$$= \mu . 1 = \mu$$

We know that $\frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-z^2} dz = \lim_{z \to \infty} \int_0^{z} \frac{2}{\sqrt{\pi}} e^{-z^2}$

$= 1.$