Q2. $f(x) = \frac{2}{e^x+1}$

So, $g(f(x)) = x$

$\Rightarrow f(x) = g^{-1}(x)$

$f(x) = y$

$\Rightarrow \frac{2}{e^x+1} = y = f(x)$

$\Rightarrow ye^x + y = 2$

$\Rightarrow ye^x = 2-y$

$\Rightarrow \ln y + \ln e^x = \ln(2-y)$

$\Rightarrow \ln y + x = \ln(2-y)$

$\Rightarrow x = \ln(2-y) - \ln y$

$\Rightarrow x = \ln \frac{2-y}{y}$

$\Rightarrow x = \ln\left(\frac{2}{y} - 1\right)$

$\Rightarrow \cancel{f^{-1}(y) = \ln\left(\frac{2}{y} - 1\right)}$

$\Rightarrow \cancel{f^{-1}\left(\frac{2}{e^x+1}\right) = \ln\left(\frac{2}{y} - 1\right)}$

$\Rightarrow \cancel{g(f(x))} =$

So, $x = f^{-1}(y)$ iff $y = f(x)$

We swap the role of x and. $y = f^{-1}(x) = \ln\left(\frac{2}{x} - 1\right)$ is the inverse fnc.

On the inverse of a fnc

A fnc "f" is invertible if and only if the fnc is one-to-one and Onto both.

one-to-one → injective

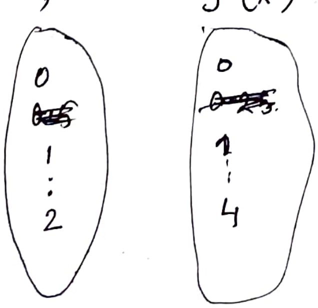
Onto → Surjective

→ Bijective / one-to-one correspondence.

However, we also can invert a function considering the range of the given fnc "f" as the domain for the inverted fnc (f^{-1}).

For instance, $f(x) = x^2$, where $f: \underbrace{(0,\infty)}_{\text{Domain}} \to \underbrace{(0,\infty)}_{\text{Co-Domain}}$

x is integer.



Domain   Co-Domain

Range: $\{0, 1, 4, \ldots\}$

→ We can take the inverse with respect to this Range.

So, $f(x) = x^2 = y$. So, $y = x^2$

$\Rightarrow x = \pm\sqrt{y}$

$\Rightarrow x = \sqrt{y}$ [as it is non-negative as per the definition]

We swap their role. $y = \sqrt{x} = f^{-1}(x)$

So, $y = f^{-1}(x) = \sqrt{x}$

with Domain of $f^{-1}(x)$ = Range of f = $\{0, 1, 4, \dots\}$

Important: So, invertible fn^c can be considered for one-to-one / injective fn^c. Thus, the question arises –

How to check if a fn^c is a one-to-one fn^c ?

Checking One-to-one

- A fn^c is one-to-one if for two different domain elemen x_1 and x_2, the following equation that assures distinct images for x_1 and x_2

$$f(x_1) = f(x_2) \text{ then } x_1 = x_2$$

Given, $f(x) = \dfrac{x+5}{x-6}$, Is it one-to-one ?

$$f(x_1) = \frac{x_1+5}{x_1-6} = f(x_2) = \frac{x_2+5}{x_2-6}$$

$$\Rightarrow (x_1+5)(x_2-6) = (x_1-6)(x_2+5)$$

$$\Rightarrow \cancel{x_1x_2} - 6x_1 + 5x_2 - \cancel{30} = \cancel{x_1x_2} + 5x_1 - 6x_2 - \cancel{30}$$

$$\Rightarrow -11x_1 = -11x_2$$

$$\Rightarrow x_1 = x_2$$

- We also can use the concept of increasing and decreasing fn' to identify if a fn' is one-to-one or not.

  Specifically, a strictly increasing and a strictly decreasing fn' - both are one-to-one fn'.

So, if a fn' f(x) has the derivative calculated for all x is greater than zero (0) then the fn' is increasing, that is, it is one-to-one as well.

Example: $f(x) = xe^{x^4}, \ \forall x \in \mathbb{R}$

$$\Rightarrow f'(x) = e^{x^4} \frac{d}{dx} x + x \frac{d}{dx} e^{x^4}$$

$$= e^{x^4} + x e^{x^4} \cdot \frac{d}{dx} x^4$$

$$= e^{x^4} + x e^{x^4} \cdot 3x^3$$

$$= e^{x^4} + 3x^4 e^{x^4}$$

As $f'(x) > 0$, for all $x \in \mathbb{R}$, we can say that the fn' is always increasing.

Q. Is the function $f(x) = |x|$ one-to-one? $x \in \mathbb{R}$.

$f(1) = |x| = 1$ So, not one-to-one.

$f(-1) = |-1| = 1$

However, if it is defined $f(x) = |x|, \ \forall x \in \mathbb{R}^+$

then it is one-to-one

Taylor's Series

Consider a polynomial — say, $f(x) = a_0 + a_1 x + \dots a_n x^n$

Polynomial

x → involves only non-negative integers powers of x.

Generally, A $\boxed{\text{polynomial}}$ of degree "n" is a function that has the below form:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots \dots + a_n x^n$$

- real numbers
- known as coefficients

For instance, $f(x) = 4x^3 - 3x^2 + 2$

- Polynomial of degree 3

$f(x) = x^7 - 4x^5 + 1$ degree ?

Answer: 7

⊞ Degree of the polynomial means the highest order of x.

Highest Power of x	Degree of Polynomial	Name
x^0	0	Constant
x^1	1	Linear
x^2	2	quadratic
x^3	3	Cubic
x^4	4	Quartic

Q. The fn^c $f(x) = 0$, is it a polynomial ?

Answer : Yes, it's a polynomial. But.
it's degree is "undefined"

Roots of polynomials

Given a polynomial $f(x) = (x-a)(x-b)$, we can find the roots as below: $f(x) = 0$

$$\Rightarrow (x-a)(x-b) = 0$$

$$\Rightarrow x = a, x = b$$

Sometimes, roots could be repeated. For instance,

$$f(x) = (x-2)^2$$

Roots are, $f(x) = 0 \Rightarrow (x-2)(x-2) = 0$

$$\Rightarrow x = 2, 2$$

Another polynomial, $f(x) = (x-2)^3 (x+4)^4$

so, roots are $\left[\begin{array}{l} x = 2, 2, 2 \\ x = -4, -4, -4, -4 \end{array}\right.$

Repeated roots

Here, root 2 has multiplicity 3
root -4 has multiplicity 4

Multiplicity of Roots :

If the multiplicity of roots is known, it provides information on the sketching of the given fn^c.

For instance, Let's say, we have a polynomial

$$f(x) = (x-2)^2(x+1)$$

Roots : 2, −1

Multiplicity 2

Here, maximum power of x is 3.

Shape of the graph

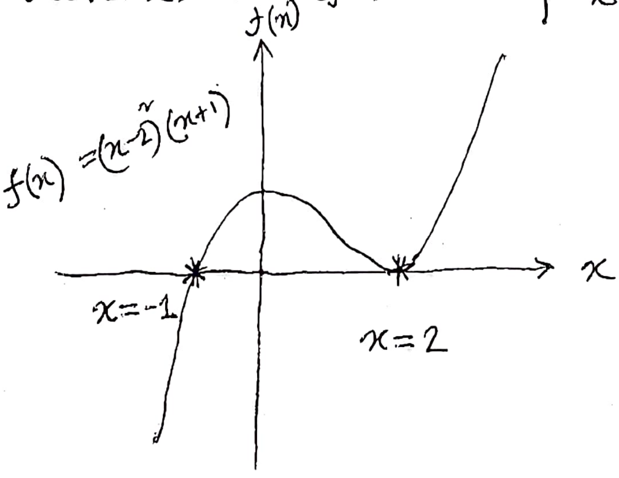
$$(x^2-4x+4)(x+1)$$
$$= (x^3-4x^2+\cancel{4x}+x^2-\cancel{4x}+4)$$
$$= (x^3-3x^2+4)$$

Positive coefficient

So, curve increases to right and decreases to left.

Because the root "2" has multiplicity 2 (which is even), the graph just <u>touches</u> x-axis.

The root "−1" has odd multiplicity. So, the graph crosses the x-axis.



So, multiplicity provides information whether a graph touches or intersects the corresponding axis.

Ex: $f(x) = (x-3)^2(x+1)^5(x-2)^3(x+2)^4$

Root	Multiplicity	Touches/ Crosses x-axis.
3	2	Touches
−1	5	Crosses
2	3	Crosses
−2	4	Touches

Power Series

Let's consider a special type of infinite series, namely the power series, which has the following form:

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

for real numbers a_n, and c. Here, x is the parameter.

A number of fn^c can be represented as power series: For instance,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n .$$

smooth fn^c

Generally, if a fn^c is infinitely differentiable, it can be represented as a power series of the form

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

$$= c_0 + c_1 (x-a) + c_2 (x-a)^2 + c_3 (x-a)^2 + \cdots$$

$\Rightarrow f(a) = c_0$

Now, as $f(x)$ is differentiable for infinitely large times

so, $$f'(x) = c_1 + 2c_2 (x-a) + 3c_3 (x-a)^2 + \cdots \cdots$$

$$\Rightarrow f'(a) = c_1$$

$$f''(x) = 2c_2 + 3.2.c_3(x-a) + \dots \dots$$

$$\Rightarrow f(a) = 2c_2 \Rightarrow c_2 = \frac{f''(a)}{2} = \frac{f''(a)}{2!}$$

$$f'''(x) = 3.2.c_3 + 3.4.c_4(x-a)^2 + \dots \dots$$

$$\Rightarrow f'''(a) = 6.c_3 \Rightarrow c_3 = f'''(a)/6 = \frac{f'''(a)}{3!}$$

Thus, we obtain, in general,

$$f^n(a) = n!\,c_n$$

$$\Rightarrow c_n = \frac{f^n(a)}{n!}$$

Taylor Series: if a fn^c is derivativable infinitely at a point "a" the fn^c and its Taylor series are equal at the neighborhood of the point "a".

Taylor series:

Let's assume that $f(x)$ has a power series expansion at $x=a$ with radious of convergence $R>0$, then the series expansion of $f(x)$ has the below form:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + f'(a).(x-a) + \frac{f''(a).(x-a)^2}{2!} + f'''(a).\frac{(x-a)^3}{3!} + \dots \dots$$

When $a=0$, we get Maclaurin Series

▣ Find Taylor's series for $f(x) = e^x$ at $x=0$

$$f(x) = f(a) + f'(a).(x-a) + \frac{f''(x-a)^2}{2!} + \frac{f'''(x-a)^3}{3!} + \dots\dots$$

Here, $f(x) = e^x$ | Here, $a = 0$

$\Rightarrow f(0) = e^0 = 1$

$f'(x) = e^x$ | $f''(x) = e^x$ | $f'''(x) = 0$

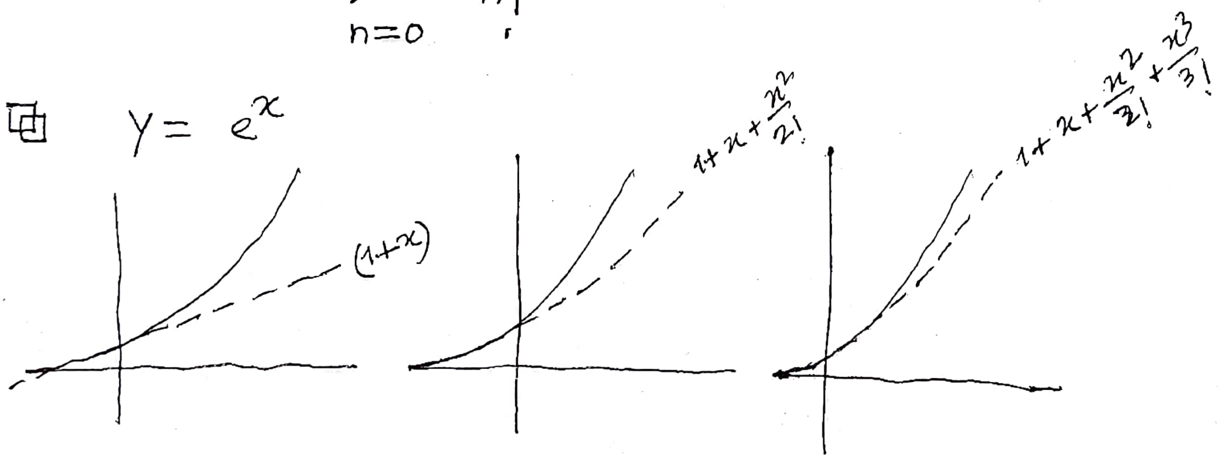
$\Rightarrow f'(0) = 1$ | $\Rightarrow f''(0) = 1$ | $\Rightarrow f'''(0) = 1$

So,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\dots$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

▣ $y = e^x$



⟶ error reducing w.r.t number of terms being considered.

▣ Find Taylor series of $\sin x = f(x)$
$\cos x = g(x)$.

$f(x) = \sin x$. Let's approximate $f(x)$ at the x value to be "0". We employ Taylor's series for this. By defn.

$$f(x) \approx f(0) + f'(0)\,\frac{x-0}{1!} + f''(0)\,\frac{(x-0)^2}{2!} + \frac{f'''(0)(x-0)^3}{3!} + \cdots\cdots$$

where. $f'(0)$ is the 1^{st} derivative calculated at $x=0$.

f'', f''' are 2^{nd} and 3^{rd} derivative, respectively

So.

$$f(x) = \sin x \qquad \Rightarrow f'(x) = \cos x \qquad \Rightarrow f'(x)\Big|_{x=0} = 1$$

$$f''(x) = -\sin x \qquad \Rightarrow f''(x)\Big|_{x=0} = \cancel{\cos x} = 0$$

$$f'''(x) = -\cos x \qquad \Rightarrow f'''(x)\Big|_{x=0} = -1$$

Thus,

$$f(x) = f(0) + f'(0)\,\frac{(x-0)}{1!} + f''(0)\,.\,\frac{(x-0)^2}{2!} + \frac{f'''(0)\,.\,(x-0)^3}{3!} + \cdots\cdots$$

$$= 0 + 1\,.\,\frac{x}{1!} + 0\,.\,\frac{x^2}{2!} + (-1)\,.\,\frac{x^3}{3!} + \cdots\cdots$$

$$= x - \frac{x^3}{3!} = x - \frac{x^3}{6}$$

We can also accomodate higher order terms $\quad f^{IV}(x) = \sin(x) \Rightarrow f^{IV}(x)\Big|_{x=0} = 0$

$$f^{V}(x) = \cos(x) \Rightarrow f^{V}(x)\Big|_{x=0} = 1$$

$$f^{VI}(x) = \sin(x) \Rightarrow f^{VI}(x)\Big|_{x=0} = 0$$

The approximation can be extended further —

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\cdots$$

Geometric Sequence : The terms are of the form

$ar^0, ar^1, ar^2, \dots \dots ar^n, ar^{n+1}$

where

$$\frac{ar}{ar^0} = \frac{ar^2}{ar^1} = \dots = \frac{ar^{n+1}}{ar^n} = r \text{ (Known as the common ratio)}$$

Sum of a finite geometric series of power n is

$$a + ar + ar^2 + \dots + ar^n = \sum_{K=0}^{n} ar^k = \frac{ar^{n+1} - a}{r-1}, \quad r \neq 1$$

$$= \frac{a\left(r^{n+1} - 1\right)}{r-1}$$

Where a : 1^{st} term

r : common ratio

$\boxed{\text{When } |r| < 1 :}$ Consider that we have an infinite geometric series. That is $n \to \infty$. So, any fraction less than 1, if raised to the power n, approaches to 0 as $n \to \infty$. For instance, $r = 0.5$

So,

$r^1 = 0.5$

$r^2 = 0.25$

$r^3 = 0.\cancel{1}125$

$\vdots$

$r^{10} = 0.000976$

$\vdots$

≈ 0

Thus.

$$\lim_{n \to \infty} \frac{a\left(r^{n+1} - 1\right)}{r-1}$$

$$= \frac{-a}{r-1} = \frac{a}{1-r} \quad \underline{\text{Ans}}$$

Q. What happens if $r > 1$?

Considering the sum again

$$\sum_{K=0}^{n} ar^k = \frac{ar^{n+1} - a}{r-1}, \quad r \neq 1$$

Here the term r^{n+1} keeps increasing as n is increasing for $r > 1$. So, the term r^{n+1} becomes ∞

So, "NO"

So, if we consider $a = 1$, we obtain for the geometric sequence $\{1, r, r^2, \dots r^n\}$

$$\sum_{K=0}^{n} r^K = \frac{a\left(r^{n+1} - 1\right)}{r-1}, \text{ where } a = 1.$$

$$= \frac{r^{n+1}-1}{r-1} = \frac{1-r^{n+1}}{1-r}$$

When, infinite sequence.

$$\sum_{K=0}^{\infty} r^K = \frac{1}{1-r}$$

□ $\displaystyle\sum_{K=2}^{\infty} \frac{1}{2^K} = ?$

$$= \frac{1}{4} + \frac{1}{2^3} + \frac{1}{2^4} + \dots$$

$$= \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$

$$= \frac{1}{4}\left(1 + \frac{1}{2} + \frac{1}{4} + \dots\right)$$

$$= \frac{1}{4}\left(1 + \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \dots\right)$$

$= \frac{1}{4} \cdot \frac{1}{1-r}$, where $r = 1/2$ and as it is infinite geometric sequence.

$$= \frac{1}{4} \cdot \frac{1}{1-\frac{1}{2}} = \frac{1}{4} \times 2 = \frac{1}{2} \quad \text{Ans}$$

□ $\displaystyle\sum_{K=3}^{\infty} \frac{1}{2^K} = ?$

Similarly, $\displaystyle\sum_{K=3}^{\infty} \frac{1}{2^K} = \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots \dots$

$$= \frac{1}{8}\left(1 + \frac{1}{2} + \frac{1}{4} + \dots \dots\right)$$

$$= \frac{1}{8}\left(1 + \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \dots\right)$$

$$= \frac{1}{8} \times \frac{1}{1-r}$$

Q. What is the sum of the sequence $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \dots\}$?

$$\sum_{K=1}^{\infty} \frac{1}{K} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots = \infty$$

□ ~~Corollar~~ Corollary: Consider that $r > 0$ and $r < 1$, then

$$\sum_{K=1}^{\infty} K r^{K-1} = 1 + 2r + 3r^2 + 4r^3 + \dots + \dots$$

$$= \frac{1}{(1-r)^2}$$

Proof: We know that

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r} = 1 + r + r^2 + \cdots + \cdots$$

By taking derivative w.r.t "r" we obtain

$$\frac{d}{dr}\sum_{k=0}^{\infty} r^k = \sum_{k=0}^{\infty} \frac{d}{dr} r^k = \sum_{k=1}^{\infty} k r^{k-1} = L.H.S$$

At R.H.S, $\frac{d}{dr}\left(\frac{1}{1-r}\right) = \frac{d}{dr}(1-r)^{-1} = -1(1-r)^{-1-1} \cdot \frac{d}{dr}(1-r)$

$$= \frac{-1}{(1-r)^2} \times (-1)$$

$$= \frac{1}{(1-r)^2}$$

So, $$\sum_{k=1}^{\infty} k r^{k-1} = \frac{1}{(1-r)^2}$$

Proved

Consider $$\frac{d}{dr}\sum_{k=0}^{\infty} r^k = \frac{d}{dr}(1 + r + r^2 + \cdots + \cdots) = 0 + 1 + 2r + 3r^2 + \cdots \cdots$$

$$= 1 + 2r^1 + 3 \cdot r^2 + \cdots \cdots$$

$$= 1 \cdot r^{1-1} + 2 \cdot r^{2-1} + 3 \cdot r^{3-1} + \cdots$$

$$= \sum_{k=1}^{\infty} k r^{k-1}$$

Calculate $$\sum_{k=1}^{\infty} k \cdot \frac{2}{3^k} = ?$$

$$\sum_{k=1}^{\infty} k \cdot \frac{2}{3^k}$$

$$= 1 \times \frac{2}{3} + 2 \cdot \frac{2}{3^2} + 3 \cdot \frac{2}{3^3} + \cdots \cdots \cdots$$

$$= \frac{2}{3}\left(1 + 2 \cdot \frac{1}{3} + 3 \cdot \frac{1}{3^2} + \cdots \cdots\right)$$

where $r = \frac{1}{3}$

$$= \frac{2}{3}\left(1 + 2 \cdot r + 3 \cdot r^2 + \cdots \cdots\right)$$

$$= \frac{2}{3} \cdot \frac{1}{(1-r)^2} = \frac{2}{3} \cdot \frac{1}{(1-\frac{1}{3})^2} = \frac{2}{3} \times \frac{9}{4} = \frac{3}{2}$$ Ans

Permutations & Combinations

Example:

How many poker hands of five cards can be dealt from a standard deck of 52 cards ?

Also, how many ways are there to select 47 cards from a standard deck of 52 cards.

$$C(52,5) = \frac{52!}{47!\,5!} = 2,598,960$$

We create combination of five from the the available poker pool.

- We use $C(n,r) = \dfrac{n!}{r!\,(n-r)!}$

Part 2:

Second part means the same — we have to find the number of ways to select 47 cards from a set of 52 cards.

$$C(52,47) = \frac{n!}{r!\,(n-r)!} = \frac{52!}{47!\,(52-r)!}$$

$$= \frac{52!}{47!\,5!}$$

Corollary:

Let n and r be nonnegative integers with $r \leq n$. Then,

$$C(n,r) = C(n, n-r)$$

Proof:

$$C(n, r) = \frac{n!}{r!\,(n-r)!}$$

and

$$C(n, n-r) = \frac{n!}{(n-r)!\,(n-(n-r))!}$$

$$= \frac{n!}{(n-r)!\,(n-n+r)!} = \frac{n!}{(n-r)!\,(r!)}$$

A group of 30 people have been trained as astronauts to go on the first mission to Mars.

How many ways are there to select ~~five people~~ a crew of six people to go on this mission.

Answer: Number of ways to select six crews from the available pool would be combination problem. So, number of such combinations would be:

$$C(30, 6) = {}^{n}C_{r} = \frac{n!}{(n-r)!\,r!} = \frac{30!}{24!\,6!}$$

$$= \frac{30 \times 29 \times 28 \times 27 \times 26 \times 25}{6!}$$

<u>Answer</u>

[4] How many permutations of the letters ABCDEFG contain:

ⓐ the string BCD

$\overset{1}{\overline{BCD}}$ $\overset{2}{\bullet}$ $\overset{3}{\bullet}$ $\overset{4}{\bullet}$ $\overset{5}{\bullet}$

So, it should be $5! = 120$

ⓑ the string CFGA

$\overset{1}{\overline{CFGA}}$ $\bullet \bullet \bullet$ $4! = 24$

ⓒ strings BA and GF

$\overset{1}{\overline{BA}}$ $\overset{2}{\overline{GF}}$ $\overset{3}{\bullet}$ $\overset{4}{\bullet}$ $\overset{5}{\bullet}$ $5! = 120$

ⓓ strings ABC and DE

$\overset{1}{\overline{ABC}}$ $\overset{2}{\overline{DE}}$ $\overset{3}{\bullet}$ $\overset{4}{\bullet}$ $4! = 24$

ⓔ the strings ABC and CDE

$\underbrace{ABCDE}_{1}$ $\underset{2}{\bullet}$ $\underset{3}{\bullet}$

Answer: $3! = 6$

ⓕ the strings CBA and BED

Two B's in different locations are not possible. So, Answer is zero.

Examples:

Q. How many ways are there to select a first-prize winner, a second-prize winner and a third-prize winner from a pool of 100 contestant ?

Answer: Here, anyone could be 1st-prize winner
2nd-prize winner
3rd-prize winner

So, it is a permutation count problem. There are

$$ {}^{100}P_3 = \frac{100!}{(100-3)!} = \frac{100!}{97!} = 100 \times 99 \times 98 $$

Ans.

Alternatively, The problem can be done using the product rule.

Task

1st Prize	2nd prize	3rd prize
n_1 ways	n_2 ways	n_3 ways
100	99	98

As per the product rule: $100 \times 99 \times 98$

r-combination

By definition, the number of r-combination of a set of n distinct elements is denoted as

$$^nC_r \text{ or } C(n,r) \text{ or } \binom{n}{r}$$

and defined as

$$^nC_r = \frac{n!}{(n-r)!\,r!}$$

Interestingly, ^nC_r is also known as Binomial coefficient

↳ coefficient of binomial expression $(a+b)^n$

The Binomial Theorem

A binomial expression is a sum of two terms. Consider that x and y are variables and n be a nonnegative integer $(n \geq 0)$, then the binomial theorem states that

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots$$

$$+ \binom{n}{n-1}x^1 y^{n-1} + \binom{n}{n}x^0 y^n$$

$$= \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$$

Q. What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$?

$$(2x-3y)^{25} = (2x+(-3y))^{25}$$

Let's assume
$2x = A$
$-3y = B$

$$= (A+B)^{25}$$

$$= \sum_{i=0}^{25} (\overset{?}{A}+B)^{25}$$

So, the coefficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$ is

$$\binom{25}{13} 2^{12} (-3)^{13} = \binom{25}{13} 2^{12} 3^{13} (-1)$$

$$= \frac{25!}{(25-13)!\ 13!} 2^{12} 3^{13} (-1)$$

$$= \frac{25!}{12!\ 13!} 2^{12} 3^{13} (-1)$$

Prove $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

$$\text{R.H.S} = (1+1)^n$$

$$= \binom{n}{0} 1^n 1^0 + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} 1^0 1^n$$

$$= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots \cdots + \binom{n}{n}$$

$$= \sum_{k=0}^{n} \binom{n}{k}$$

Q. Prove $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$

$$L.H.S = \sum_{k=0}^{n} (-1)^k \binom{n}{k} = (-1)^0 \binom{n}{0} + (-1)^1 \binom{n}{1} + (-1)^2 \binom{n}{2} + \cdots + (-1)^n \binom{n}{n}$$

$$= \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \binom{n}{4} - \binom{n}{5} + \cdots (-1)^n$$

$$\vdots$$

$$= 0$$

$$\Rightarrow \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$$

We can expand. $((-1)+(1))^n$

$= 0^n = ((-1)+(1))^n$ = we can expand as shown above

Q. Prove that $\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$

$$R.H.S = 3^n$$
$$= (1+2)^n = \sum_{k=0}^{n} 1^{n-k} \, 2^k \cdot \binom{n}{k} = \sum_{k=0}^{n} 2^k \binom{n}{k}$$

As, n is an integer
k is an integer

So, n-k is an integer

Examples

Q. What is the expansion of $(x+y)^4$

$$(x+y)^4 = \sum_{i=0}^{4=n} \binom{n}{i} x^{4-i} y^{i} = \sum_{i=0}^{4} \binom{4}{i} x^{4-i} y^{i}$$

$$= \binom{4}{0} x^{4} y^{0} + \binom{4}{1} x^{4-1} y^{1} + \binom{4}{2} x^{4-2} y^{2} + \binom{4}{3} x^{4-3} y^{3} + \binom{4}{4} x^{4-4} y^{4}$$

$$= x^4 + \frac{4!}{3!\,1!} x^3 y + \frac{4!}{2!\,2!} x^2 y^2 + \frac{4!}{3!\,1!} x^1 y^3 + x^0 \cdot y^4$$

$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

Q. What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x+y)^{25}$?

Answer: We use the binomial coefficient concept. $^{n}C_{r}$ or $C(n, r)$ or $\binom{n}{r}$

Here, r denotes the power of y

$n-r$ denotes the power of x

So, $\binom{25}{13}$ is the coefficient

$$= \frac{25!}{(25-13)!\,13!} = \frac{25!}{12!\,13!}$$ Ans

Pascal's Identity and Triangle

Consider that n and k are positive integers and $n-k \geq 0$. Then,

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

$$RHS = \binom{n}{k-1} + \binom{n}{k}$$

$$= \frac{n!}{(n-(k-1))!(k-1)!} + \frac{n!}{(n-k)!\,k!}$$

$$= \frac{n!}{(n-k+1)!(k-1)!} + \frac{n!}{(n-k)!\,k!}$$

$$= \frac{n!\quad k}{(n+1-k)!\,k!} + \frac{n!\,(n-k+1)}{(n-k+1)\,(n-k)!\,k!}$$

$$= \frac{n!\quad k}{(n+1-k)!\,k!} + \frac{n!\,(n-k+1)}{(n+1-k)!\,k!}$$

$$= \frac{n!}{(n+1-k)!\,k!}\left[k + (n-k+1)\right]$$

$$= \frac{n!\,(n+1)}{(n+1-k)!\,k!} = \frac{(n+1)!}{(n+1-k)!\,k!}$$

$$= \binom{n+1}{k} = L.H.S$$

Proved

Pascal's Triangle

$$
\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} \\
 & & & & \vdots & & & &
\end{array}
$$

$$
\begin{array}{ccccccccc|l}
 & & & & 1 & & & & & (a+b)^0 \\
 & & & 1 & & 1 & & & & (a+b)^1 \\
 & & 1 & & 2 & & 1 & & & (a+b)^2 \\
 & 1 & & 3 & & 3 & & 1 & & (a+b)^3 \\
1 & & 4 & & 6 & & 4 & & 1 & (a+b)^4 \\
 & & & & & & & & & \vdots
\end{array}
$$

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

For, $n=3$,
$k=2$

$$\binom{4}{2} = \binom{3}{1} + \binom{3}{2}$$

III Substitution

$$\int 2x \cos(x^2)\, dx$$

Integration is equivalently an anti-derivative process. So, if we can find a fn^c whose derivative is $2x\cos(x^2)$, then we can easily obtain the solution of

$$\int 2x \cos(x^2)\, dx.$$

Let's consider $f(x) = \sin(x^2)$

$$\Rightarrow \frac{d}{dx} f(x) = \cos(x^2) \frac{d}{dx} x^2$$

$$= \underbrace{2x\cos(x^2)}_{F(x)}$$

$$\Rightarrow f'(x) = F(x)$$

$$\int \frac{d}{dx} f(x)\, dx = f(x)$$

$$= \sin(x^2) + \underbrace{c}_{\text{integration constant}}$$

III However, the process is not always so straight forward. For instance,

$$\int x^3 \sqrt{1-x^2}\, dx = ?$$

It is difficult to predict a fn^c whose derivative takes the shape as

$$x^3\sqrt{1-x^2}$$

We can write

$$x^3\sqrt{1-x^2} = \underbrace{(-2x)}_{\frac{d}{dx}(1-x^2)}\left(\frac{-1}{2}\right)\left(1-\underbrace{(1-x^2)}_{z}\right).\sqrt{\underbrace{1-x^2}_{z}}$$

How?

We took a fn^c $-\frac{1}{2}(1-z)\sqrt{z}$, where $z = 1-x^2$

and multiplied using "$-2x$" the derivative of $(1-x^2)$

So, we need a fn^c whose derivative would be of the form $(-\frac{1}{2})(1-x)\sqrt{x}$

$z = 1-x^2$

$$\frac{d}{dx} F(\cancel{1-x^2}) = \frac{d}{dx} F(z)$$

$$= -2x\, F'(1-x^2)$$

Replacing x by $1-x^2$

$$= -2x\left(-\frac{1}{2}\right)\left(1-(1-x^2)\right)\sqrt{1-x^2}$$

$$= x^3\sqrt{1-x^2}$$

Thus,

$$\int -\frac{1}{2}(1-x)\sqrt{x}\,dx = \int -\frac{1}{2}(1-x)x^{\frac{1}{2}}\,dx$$

$$= \int -\frac{1}{2}x^{\frac{1}{2}} + \frac{1}{2}x^{\frac{3}{2}}\,dx$$

$$= -\frac{1}{2}\left(\frac{2}{3}x^{\frac{3}{2}} - \frac{2}{5}x^{5/2}\right) + c$$

$$= \frac{1}{5}x^{5/2} - \frac{1}{3}x^{\frac{3}{2}} + c$$

Finally

$$\int x^3\sqrt{1-x^2}\,dx = \frac{1}{5}(1-x^2)^{5/2} - \frac{1}{3}(1-x^2)^{3/2} + c$$

Ans.

However we can simplify the process of finding the candidate fn^c. For the above problem.

if we be able to sense that a given fn^c of x is the derivative of another fn^c, through chain-rule of the form $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$ then we assume that $u = f(x)$ and write the given fn^c in terms of u, with no x remaining in the given fn^c.

Following this we obtain a new integration where we integrate a f^n of u;

Replace the expression for u in terms of x.

Antiderivative of the original given f^n

Following the above steps.

$$\int x^3\sqrt{1-x^2}\,dx$$

Assume $1-x^2=u$

$\Rightarrow \frac{du}{dx}=-2x$

$$= \int x^3\sqrt{u}\cdot\frac{-2x}{-2x}\,dx$$

$$= \int \frac{x^2}{-2}\sqrt{u}\,.(-2x)\,dx = \int \frac{x^2}{-2}\sqrt{u}\,.\frac{du}{dx}dx$$

$$= \int \frac{x^2}{-2}\sqrt{u}\,du =$$

As, $1-x^2=u$

$\Rightarrow -x^2=u-1$

$\Rightarrow x^2=1-u$

Using these values, we obtain.

$$\int \frac{(1-u)}{-2}\cdot\sqrt{u}\,du$$

$$= \int -\frac{1}{2}(1-u)\cdot u^{1/2}\,du$$

$$= \int\left(-\frac{1}{2}u^{\frac{1}{2}}+\frac{1}{2}u^{3/2}\right)du$$

$$= \int -\frac{1}{2}u^{\frac{1}{2}}du + \frac{1}{2}\int u^{3/2}du$$

$$= -\frac{1}{2}\times\frac{2}{3}u^{\frac{3}{2}}+\frac{1}{2}\,\frac{2}{5}\,u^{5/2}+C$$

$$= u^{3/2}\left(\frac{1}{5}u-\frac{1}{3}\right)+C$$

Ans

⊞ Method of Partial Fractions

↳ We can split ~~the~~ a fraction into sum of fractions with comparatively simpler denominators

Given $\dfrac{5x-3}{(x+1)(x-3)} = \dfrac{A}{(x+1)} + \dfrac{B}{(x-3)}$

$\Rightarrow \quad 5x-3 = A(x-3) + B(x+1)$

When $x=3$, we get $B(3+1) = 4B = 5\times 3 - 3 = 12$

$\Rightarrow \quad B = 3$

When $x=-1$, we get $B(-1+1) + A(-1-3) = 5(-1)-3$

$\Rightarrow \quad -4A = -5-3 = -8$

$\Rightarrow \quad A = 2$

So, $\dfrac{5x-3}{(x+1)(x-3)} = \dfrac{2}{(x+1)} + \dfrac{3}{(x-3)}$

⊞ $\dfrac{-2x+4}{(x^2+1)(x-1)^2} = ? = \dfrac{Ax+B}{x^2+1} + \dfrac{C}{x-1} + \dfrac{D}{(x-1)^2}$

$\Rightarrow \quad -2x+4 = (Ax+B)(x-1)^2 + C(x^2+1)(x-1) + D(x^2+1)$

$= (Ax+B)(x^2-2x+1) + C(x^3-x^2+x-1) + D(x^2+1)$

$= A(x^3-2x^2+x) + B(x^2-2x+1) + C(x^3-x^2+x-1) + D(x^2+1)$

$= (A+C)x^3 + x^2(B-C+D-2A) + (A-2B+C)x + (B-C+D)$

By equating the coefficient of A, B, C, D in the L.H.S & R.H.S

1. $A+C=0$
2. $-2A+B-C+D=0$
3. $A-2B+C=-2$
4. $B-C+D=4$

$-2A = -4$

$\Rightarrow A = 2$

$A+C=0$

$\Rightarrow C = -A = -2$

$B-C+D=4$

$\Rightarrow 1+2+D=4$

$\Rightarrow D = 4-3$

$= 1$

$A-2B+C=-2$

$\Rightarrow 2-2B+(-2) = -2$

$\Rightarrow 2-2B = 0$

$\Rightarrow B = 1$

$$\int \frac{dx}{4x^2+4x+2} = ?$$

Consider

$$4x^2+4x+2 = 4(x^2+x)+2$$
$$= 4(x^2+x)+1+1$$
$$= 4\left(x^2+x+\frac{1}{4}\right)+1$$
$$= 4\left(x+\frac{1}{2}\right)^2+1$$

Let's assume

$$x+\frac{1}{2} = z$$
$$\Rightarrow dx = dz$$

So,

$$\int \frac{dx}{4x^2+4x+2} = \int \frac{dz}{4(z^2)+1} = \int \frac{dz}{4\left(z^2+\frac{1}{4}\right)}$$
$$= \frac{1}{4}\int \frac{dz}{z^2+\left(\frac{1}{2}\right)^2} = \frac{1}{4}\int \frac{dz}{z^2+(a)^2}, \text{ where } a=\frac{1}{2}$$
$$= \frac{1}{4}\tan^{-1}\frac{z}{a}+c = \frac{1}{4}\tan^{-1}\frac{x+\frac{1}{2}}{\frac{1}{2}}+c$$
$$= \frac{1}{4}\tan^{-1}\frac{\frac{2x+1}{2}}{\frac{1}{2}}+c$$
$$= \frac{1}{4}\tan^{-1}(2x+1)+c$$

where c is the constant of integration for indefinite integration.

$$\int \frac{dx}{\sqrt{2x-x^2}} = ?$$

Let's perform algebraic transformation on the denominator

$$\sqrt{2x-x^2} = \sqrt{-(x^2-2x)} = \sqrt{-(x^2-2x)+1-1}$$
$$= \sqrt{-(x^2-2x+1)+1} = \sqrt{-((x-1)^2)+1}$$

Let's assume that $(x-1) = u$. So, we obtain

$$\sqrt{-((x-1)^2)+1} = \sqrt{1-u^2}$$

Thus,

$$\int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}u+c$$
$$= \sin^{-1}(x-1)+c$$

So, we finally obtain

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{2x+1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}$$

⊞ Calculate $\int \frac{x+4}{x^3+3x^2-10x}\,dx$

For partial fraction method, let's assume

$$g(x) = x^3+3x^2-10x$$
$$f(x) = x+4$$

Here, degree of $f(x)$ is less than in the set up $\frac{f(x)}{g(x)}$

$$\frac{x+4}{x(x^2+3x-10)} = \frac{x+4}{x(x^2+5x-2x-10)} = \frac{x+4}{x\{x(x+5)-2(x+5)\}}$$

$$= \frac{x+4}{x(x+5)(x-2)} = \frac{A}{x} + \frac{B}{(x+5)} + \frac{C}{(x-2)}$$

$$\Rightarrow \quad x+4 = A(x+5)(x-2) + B(x-2)x + C(x+5)x$$

For, $x=0$, $\quad 4 = A(0+5)(-2) + 0 + 0$

$$\Rightarrow \quad A = -\frac{4}{10} = -\frac{2}{5}$$

For, $x=-5$, $\quad -1 = A\times 0 + B(-5-2)(-5) + 0$

$$\Rightarrow \quad B = -\frac{1}{35}$$

For $x=2$, $\quad 2+4 = A\times 0 + B\times 0 + C(7)\times 2$

$$\Rightarrow \quad C = \frac{6}{14} = \frac{3}{7}$$

We obtain,

$$\frac{x+4}{x(x+5)(x-2)} = \frac{A}{x} + \frac{B}{x+5} + \frac{C}{x-2} = -\frac{2}{5}\cdot\frac{1}{x} - \frac{1}{35}\cdot\frac{1}{(x+5)} + \frac{3}{7}\cdot\frac{1}{(x-2)}$$

$$\Rightarrow \int \frac{x+4}{x(x+5)(x-2)}\,dx = -\frac{5}{2}\ln|x| + \frac{3}{7}\ln|x-2| - \frac{1}{35}\ln|x+5| + C$$

Ans

Prove that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$

The above proof was demonstrated by Laplace and is very useful in Gaussian integral frequently needed in many processes.

Let's assume $I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$, where x is just the dummy variable.

$$\Rightarrow I.I = \int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-x^2}\,dx$$

$$\Rightarrow I^2 = \int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy$$

By replacing the dummy variable x to y.

$$\Rightarrow I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,\underbrace{dx\,dy}_{\text{small area in the XY cartesian plane}}$$

We transform the problem to polar coordinates by assuming

$x = r\cos\theta$

$y = r\sin\theta$

So, $r^2\cos^2\theta + r^2\sin^2\theta = x^2+y^2$

$\Rightarrow x^2+y^2 = r^2$

$dx\,dy = r\,dr\,d\theta$

Also, for r limit is 0 to ∞

for θ limit is 0 to 2π

That is we obtain

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy = \int_0^{\infty}\int_0^{2\pi} e^{-r^2}\,r\,dr\,d\theta$$

$$= \int_0^{\infty} e^{-r^2}\,r\,dr \int_0^{2\pi} d\theta$$

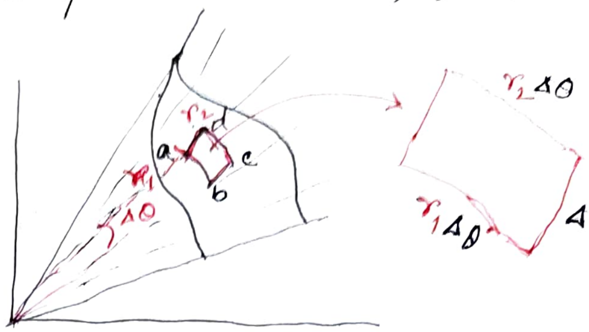
$$= \int_0^{\infty} e^{-r^2}\,r\,dr \cdot (2\pi) = 2\pi\int_0^{\infty} e^{-r^2}\,r\,dr$$

Let's assume $u = r^2$

$\Rightarrow du = 2r\,dr$

Thus, $2\pi\int_0^{\infty} e^{-u}\,du = 2\pi \times \frac{1}{2}$

The conversion of $dA = dx\,dy$ occurs as shown in the pictorial representation



$$\Delta\theta = \frac{ab}{r_1} \quad \Big| \quad \Delta\theta = \frac{cd}{r_2}$$

$$\Rightarrow ab = r_1\Delta\theta \quad \Big| \quad \Rightarrow cd = r_2\Delta\theta$$

$\Rightarrow$ considering that the dA is so infinitesimally small that we can consider that $r_1 \approx r_2$, so, let's consider $r_1 = r_2 = r$. Then, Area in the polar coordinate would be $r.\Delta\theta.\Delta r = r\Delta\theta\Delta r \approx r\,dr\,d\theta$

Calculate $\iint_D e^{x^2+y^2}\,dx\,dy = ?$

Consider, the disc-region D defined a disc of radius 2, with the disc center is centered at the origin.

Answer: Using the definition of D, we can write that the disc is defined by.

$$0 \le r \le 2 \quad : \text{radius.}$$

$$0 \le \theta \le 2\pi$$

Transformation to polar coordinate as

$$x = r\cos\theta \quad \Big| \quad x^2+y^2 = r^2$$

$$y = r\sin\theta \quad \Big| \quad dA = dx\,dy = r\,dr\,d\theta$$

Thus,

$$\iint_D e^{x^2+y^2}\,dx\,dy = \int_0^{2\pi}\!\!\int_0^2 e^{r^2}\,r\,dr\,d\theta$$

consider, $r^2 = u$

$\Rightarrow 2r\,dr = du$

$$\int_0^{2\pi}\left[\int_0^2 \frac{1}{2}e^u\,du\right]d\theta$$

$$= \int_0^{2\pi}\frac{1}{2}\left[e^2-1\right]d\theta = \frac{1}{2}\times 2\pi\left[e^2-1\right] = \pi\left[e^2-1\right]$$

Ans

So, from Laplace's solution on $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$, we can generalize

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} = \sqrt{\pi/\alpha} = \left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}}$$

Here, the variable x is ~~g~~ scaled by a factor $\sqrt{\alpha}$

$\boxed{}$ Generic form $\int_{-\infty}^{\infty} x^n e^{-\alpha x^2} dx$, for $n = 1, 2, 3 \ldots\ldots$

If n is odd, contributions from $\{-\infty, 0\}$ perfectly cancel out the contribution from $\{0, \infty\}$.

When n is even

$$\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx = \frac{\pi^{1/2}}{2\alpha^{3/2}}$$

derivative w.r.t α

$$\int_{-\infty}^{\infty} x^4 e^{-\alpha x^2} dx = \frac{\pi^{1/2}}{4\alpha^{5/2}}$$

$\vdots$

Finally we get

$$\int_{-\infty}^{\infty} x^n e^{-\alpha x^2} dx = \frac{1.3.5 \ldots \ldots (n+1)}{2^{n/2} \alpha^{(n+1)/2}}, \; n = 0, 2, 4.$$

These forms are usefull for computions involving harmonic oscillators wavefunctions.

All the evaluations, as ~~athe~~ stated earlier, are shown using the results obtained through transforming the integral into a polar coordinate problem.

However, $\int_{-\infty}^{\infty} e^{-x^2} dx$ can be calculated using the cartesian coordinate as well.

$\boxplus$ $\int_{-\infty}^{\infty} e^{-x^2} dx = ?$ Here e^{-x^2} is an even fn^c. So, we can write

$$\int_{-\infty}^{\infty} e^{-x^2} dx = 2\int_0^{\infty} e^{-x^2} dx$$

So,

$$I = 2\int_0^{\infty} e^{-x^2} dx \Rightarrow I.I = 4\int_0^{\infty} e^{-x^2} dx \int_0^{\infty} e^{-y^2} dx$$

$$\Rightarrow I^2 = 4\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)} dx\, dy$$

Let's assume that $y = xs$

$\Rightarrow dy = x\,ds.$

$$\Rightarrow I^2 = 4\int_0^{\infty}\left(\int_0^{\infty} e^{-(x^2+x^2s^2)} x\,ds\right) dx$$

$$\Rightarrow I^2 = 4\int_0^{\infty}\left(\int_0^{\infty} e^{-x^2(s^2+1)} x\,ds\right) dx$$

$$\Rightarrow I^2 = 4\int_0^{\infty}\left(\int_0^{\infty} e^{-x^2(s^2+1)} x\,dx\right) ds$$

Using Fubini's Theorem to switch the order of the integration.

$\Rightarrow \cancel{I^2 = 4\int}$

$-x^2 = z$

$\Rightarrow -2x\,dx = dz$

$$\Rightarrow I^2 = 4\int_0^{\infty}\left(\frac{1}{-2}\int_0^{\infty} e^{+z(s^2+1)} (-2x)\, dx\right) ds$$

$$\Rightarrow I^2 = 4\int_0^{\infty}\left(\left[\frac{e^{-x^2(s^2+1)}}{-2(s^2+1)}\right]_0^{\infty}\right) ds$$

$$\Rightarrow I^2 = 4\int_0^{\infty} \frac{1}{-2(s^2+1)} ds = 4\int_0^{\infty} \frac{1}{2}\frac{1}{1+s^2} ds$$

$$= 2\left[\tan^{-1} x\right]_0^{\infty} = 2\cdot\left[\pi/2 - 0\right] = \pi$$

So, $I = \sqrt{\pi}$ Ans