Random Variables

Review: Mapping and Functions

Definition: Given two abstract spaces S and A, an A-valued function is defined as

$$f : S \longrightarrow A$$

where, "f" assigns a specific element from A to each element of S

So, assuming that $\omega \in S$, we can write

$$f(\omega) \in A, \quad \forall \omega \in S$$

Within $f : S \rightarrow A$ $\quad$ S: Domain $\rightarrow$ Range when all elements are image
A: Co-Domain

When we study real valued fn^c, the def^n appears as follows: $f : \mathbb{R} \rightarrow \mathbb{R}$ $\quad$ $S = \mathbb{R}$, $A = \mathbb{R}$

Definition: Let's consider two sets $F \subset S$ and, $G \subset A$

we define image of F under f as:

$$f(F) = \{a \in A : a = f(\omega) \text{ for some } \omega \in F\}$$

and, the pre-image or inverse-image of $G \subset A$ under "f" is defined as:

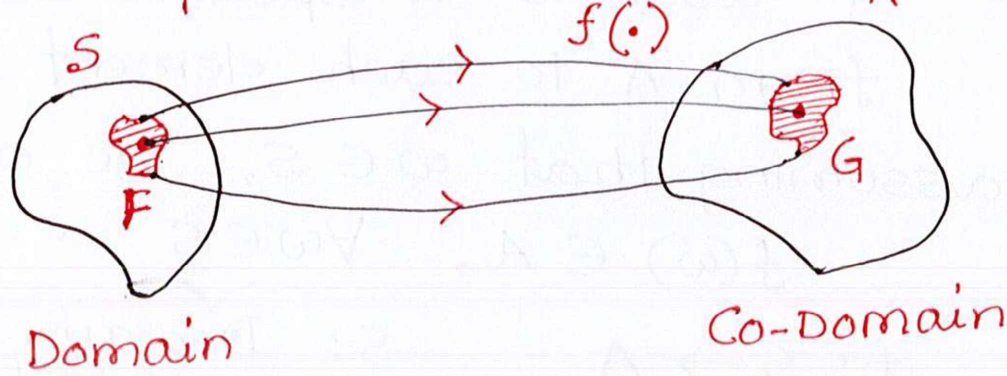
$$f^{-1}(G) = \{\omega \in S : f(\omega) \in G\}$$

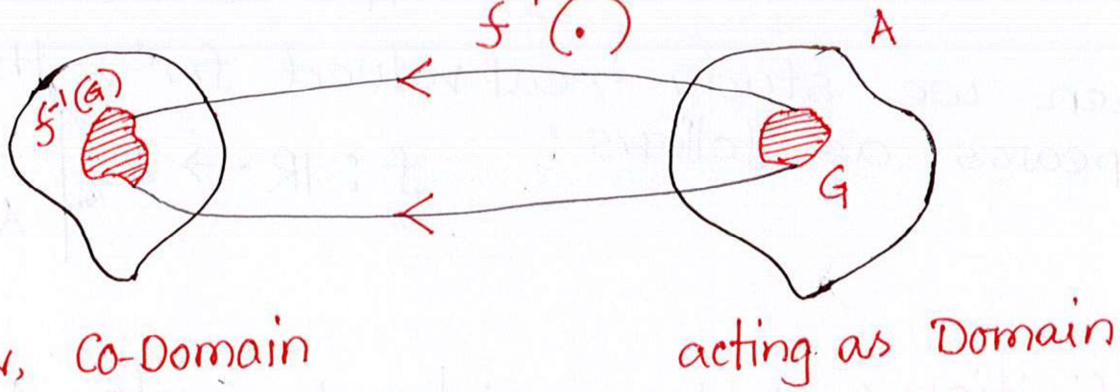
That is,

$f(F)$ is the set of all points in A obtained by mapping the points of F under the definition of "f"

$f^{-1}(G)$ is the set of all points $\omega \in S$ that map to G

Pictorial representation :





☐ Generally, in calculus we study real-valued functions of a real variable

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$

as we see, $S = \mathbb{R}$ and $A = \mathbb{R}$

Why do we study random variables

We perform experiments to measure a quantity, characterize a system. All the experiments, generally, have different form of outcomes.

So, it is common that outcomes of experiments are interpreted, and often

We may not be interested in the precise outcome, rather a modified form using functional mapping be of our interest.

generates a number

Examples of experiments:

For instance,

- Suppose, we are measuring induced electric current in antenna due to random thermal motion of charges
- Suppose, we have a coin and we perform N tosses. Now, we are interested in the number of heads in the sequence generated after N-tosses.

interestingly,

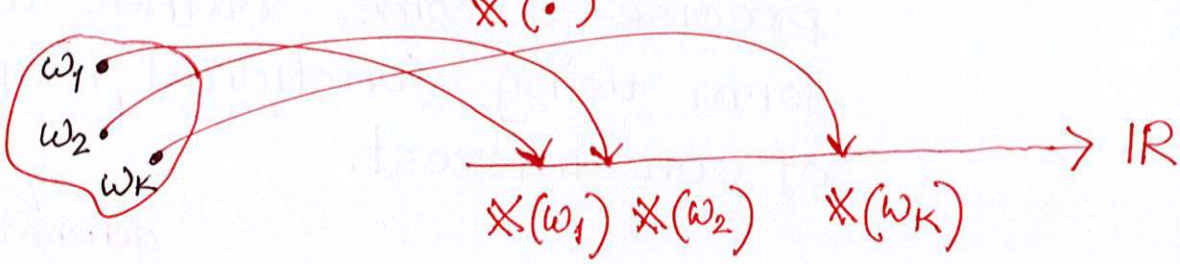
we don't consider the exact sequence

Here, each outcome produces a number from a numerical function

Intuitively,

Given, $(S, \mathcal{F}, P)$, a random variable is a mapping from the sample space S to the real line $\mathbb{R}$. If we denote random variable by X, then

$X : S \rightarrow \mathbb{R}$. The pictorial representation is as follows:



Here, $\omega_1, \omega_2 \ldots \omega_K$ are the outcomes and the corresponding numeric values are $X(\omega_1), X(\omega_2) \ldots X(\omega_K)$.

So, the numerical value becomes random

So, in a random experiment, the value of ω is chosen as random because of the possible outcome of the experiment.

Observation:

- Random variable X is a function from sample space S to the real line $\mathbb{R}$
- The term "random" indicates the underlying randomness of choosing an element ω from the sample space

- When the value of ω is fixed, the mapping function $X(\omega)$ generates a fixed value from $\mathbb{R}$

- Interestingly,

  Probability measure is associated with events, whereas

  Random variable is associated with each elementary outcome of the sample space.

In short, the randomness in the observed value $X(\omega)$ for the outcome ω is due to the randomness of $\omega \in S$ in any random experiment defined as:

$$(S, \mathcal{F}, P)$$

And, the mapping $X: S \to \mathbb{R}$ is not random. Instead, it is fixed and deterministic.

⊞ Observation:

Not all subsets of the sample space are considered as events, and the same goes for the random variable. That is, not all functions from S to $\mathbb{R}$ are considered as random variable

So, random variable X is a fn^c with traits as:

The function $X: S \rightarrow \mathbb{R}$ has the property that its output inherits a probability measure from P in the probability space defined as $(S, \mathcal{F}, P)$

Let's recall the concept of Borel set again

Given S and a family of subsets $G = \{A_i,\ i \in I\}$ of S, the σ-field (sigma-field) generated by G, denoted as $\sigma(G)$, is the smallest σ-field containing all the subsets in G.

So, when $S = \mathbb{R}$,

we need the smallest σ-field containing all the open intervals

$$(a, b),\ a < b,\ a, b \in \mathbb{R}$$

That is, we want

$$\sigma(G), \text{ where } G = \{(a,b),\ \forall a, b \in \mathbb{R} \text{ and } a < b\}$$

Interestingly,

$\sigma(G)$ contains all the countable sequences of set operations

- Union $\cup$
- Intersection $\cap$
- complement $-$

along with the open intervals

on any collection of open intervals

With understanding on σ-field, we can now define Borel field and Borel sets.

Borel field : Given the set of real numbers $\mathbb{R}$, the Borel field of $\mathbb{R}$, denoted as $B(\mathbb{R})$, is defined as the σ-field generated by open intervals

$$G = \{(a,b) : \forall a, b \in \mathbb{R} \text{ and } a < b\}$$

Borel Set : The members of $B(\mathbb{R})$ are known as Borel sets.

As the Borel field contains all the open interval, it also contains the below intervals:

$$(-\infty, b) = \bigcup_{n=1}^{\infty} (b-n, b) = \lim_{m \to \infty} (-m, b)$$

$$(a, \infty) = \bigcup_{n=1}^{\infty} (a, a+n) = \lim_{m \to \infty} (a, m)$$

$$\{a\} = \bigcap_{n=1}^{\infty} \left(a - \tfrac{1}{n}, a + \tfrac{1}{n}\right) = \lim_{n \to \infty} \left(a - \tfrac{1}{n}, a + \tfrac{1}{n}\right)$$
$$= \{a\}$$

It also follows that

$$[a, b) = \{a\} \cup (a, b) \in B(\mathbb{R})$$
$$(a, b] = (a, b) \cup \{b\} \in B(\mathbb{R})$$
$$[a, b] = \{a\} \cup (a, b) \cup \{b\} \in B(\mathbb{R})$$

In addition to all these intervals, all finite and countable sequences of set operations of unions ($\cup$), intersections ($\cap$), and complements ($^{-}$) of those intervals will be in the Borel Field $B(\mathbb{R})$.

In short,

$B(\mathbb{R})$ includes all the subsets of $S=\mathbb{R}$ that we could be interested in. One might ask, if it is sufficient or not.

Interestingly, proving that $B(\mathbb{R}) \neq \wp(\mathbb{R})$ is always difficult.

Yes, there may be some sets that are not in Borel field of $\mathbb{R}$, but those sets are strange and of no interests in probability problems.

Also, often, we don't need the whole real line $\mathbb{R}$ as the sample space. That is,

$$S = A, \text{ where } A \subset \mathbb{R}$$

example: $A=[0, 1]$

So, the event space should be the Borel field of A, that is $B(A)$

We can cut the $B(\mathbb{R})$ to obtain the $B(A)$

$$B(A) = \{ F \cap A : \forall F \in B(\mathbb{R}) \}$$

With the revision of Borel Field and Borel set, we can now explore the concept of Random Variable.

It is not random

It is not a variable

Rather, it is a fn^c that maps random outcomes "ω" to a real numeric value through a numerical fn^c.

So, random variable, defined as a fn^c, follow the mapping $X : S \to \mathbb{R}$ inherits a probability measure from P in the underlying probability space $(S, \mathcal{F}, P)$

Simply, $X(\cdot)$ should have the property that for any $A \in B(\mathbb{R})$, we can compute the probability $P_X(A)$.

And any function $X(\cdot)$ such that we can calculate $P_X(\cdot)$ for all $A \in B(\mathbb{R})$ is called a Borel measurable function

[1] Measurable function

Let's consider $(S, \mathcal{F})$ be a measurable space. Then a function $f: S \to \mathbb{R}$ is $\mathcal{F}$-measurable function if the pre-image of every Borel set A, is an $\mathcal{F}$-measurable subset of S. $A \in \mathcal{B}(\mathbb{R})$

The pre-image of A is defined as:

$$f^{-1}(A) \triangleq \{ \omega \in S \mid f(\omega) \in A \}$$

As per the definition of measurable function, $f: S \to \mathbb{R}$ is an $\mathcal{F}$-measurable fn^c if $f^{-1}(A)$ is an $\mathcal{F}$-measurable subset of S for every Borel set A.

extending the def^n for probability space.

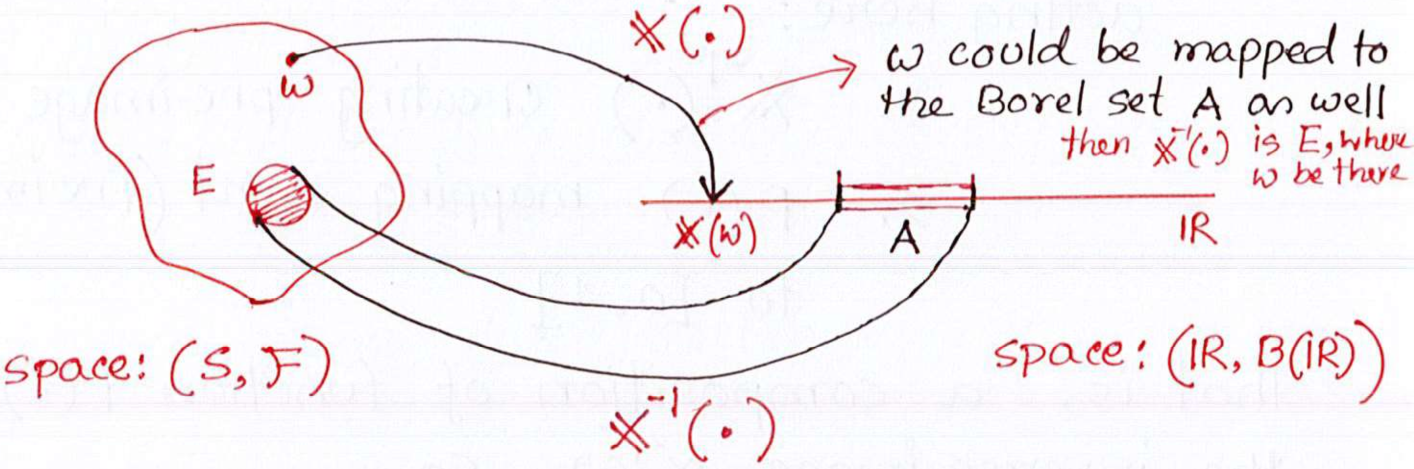
Given $(S, \mathcal{F}, P)$ a random variable X is an $\mathcal{F}$-measurable function $X: S \to \mathbb{R}$ from $(S, \mathcal{F})$ to $\mathbb{R}$, with property as follows

$$\left. \begin{aligned} X^{-1}(A) &= \{ \omega \in S \mid X(\omega) \in A \} \\ \& \ X^{-1}(A) &\in \mathcal{F} \end{aligned} \right] \text{ for all } A \in \mathcal{B}(\mathbb{R})$$

In other words, every Borel set $A \in \mathcal{B}(\mathbb{R})$ is the event E in the event space $\mathcal{F}$.

So, X is a random variable that maps every $\omega \in S$ to the real line $\mathbb{R}$.

We can represent the random variable $\mathbb{X}$ definition pictorially as:



Now, the pre-image

$$\mathbb{X}^{-1}(A) \triangleq \{\omega \in S \mid \mathbb{X}(\omega) \in A\} \in \mathcal{F}$$

indicates that $\mathbb{X}^{-1}(A)$ is an event.

means

It has the associated probability measure.

Probability Law of a random variable $\mathbb{X}$

Let's denote $P_{\mathbb{X}}$ as the probability law of the random variable $\mathbb{X}$. So,

the probability Law $P_{\mathbb{X}}$ of a random variable $\mathbb{X}$ is a function $P_{\mathbb{X}}$

$$P_{\mathbb{X}} : B(\mathbb{R}) \longrightarrow [0, 1]$$

defined as

$$P_{\mathbb{X}}(A) \triangleq (\{\omega \in S \mid \mathbb{X}(\omega) \in A\})$$
$$= P(\mathbb{X}^{-1}(A))$$

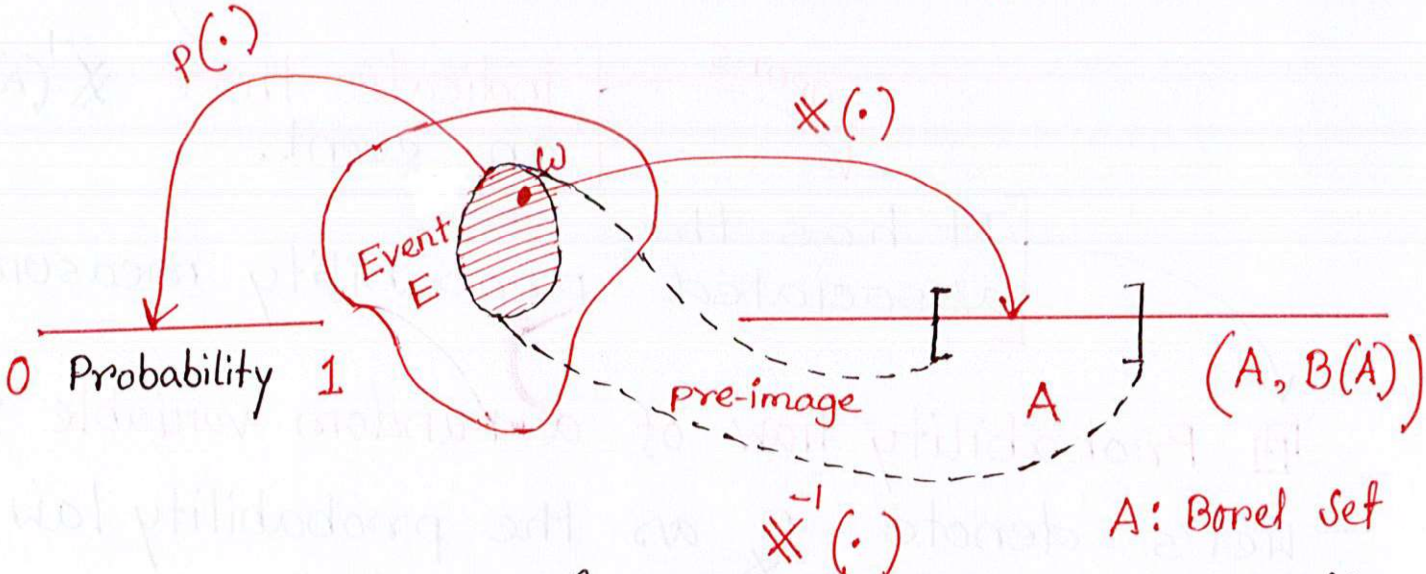
As we observe, we have two functions acting here:

1. $X^{-1}(\cdot)$ creating pre-image
2. $P(\cdot)$ mapping event (pre-image) to $[0, 1]$

That is, a composition of function $P(\cdot)$ and the inverse image $X^{-1}(\cdot)$. So,

$$P_X(\cdot) = P(\cdot) \circ X^{-1}(\cdot)$$

composition notation



With a short-hand notation, the probability law could be written as:

$$P_X = P \circ X^{-1}$$

Original probability measure

We just use the interested Borel set right next to it