

We start our discussion with

Basic Probability Ideas

and the first topic that we would like to focus on is the **SET THEORY**

↳ why?

Precisely, the mathematical basis of probability is the concept of set and set theory.

↳ For any random experiment, we have multiple possible outcomes.

↳ Set concepts provide an easy mechanism to enumerate the underlying possible outcomes of any given random experiment.

For example: consider rolling a fair die
die: has six faces
fair: appearance of each face is equally likely.

So, the possible outcomes are

$$S = \{1, 2, 3, 4, 5, 6\}$$

↳ is a set

↳ By definition, set is a collection of objects.

↳ could be names, numbers or anything.

Let's define a few events from the random experiment of rolling a fair die.

$$E_1 = \text{Outcome is odd} = \{1, 3, 5\}$$

$$E_2 = \text{Outcome is divisible by 2} = \{2, 4, 6\}$$

$$E_3 = \text{Outcome is prime} = \{2, 3, 5\}$$

As we see here,

Each of the above events can be represented by a subset of the defined set S .

also, known as sample space
set of all possible outcomes

From this experiment, we can ask

what is the probability that the outcome is odd
or, what is the probability that the outcome is prime

Another experiment: Suppose, a coin is tossed twice so, the possible outcomes are

$$S = \{HH, HT, TH, TT\}$$

	H	T
H	HH	HT
T	TH	TT

$$E_1 = \text{Head (H) appears first} = \{HH, HT\}$$

$$E_2 = \text{Tail (T) appears first} = \{TH, TT\}$$

How many events are possible?

For $S = \{1, 2, 3, 4, 5, 6\}$, distinct subsets would be $2^6 = 64$

Each subset is an event.

consider the coin toss experiment, where the fair coin is tossed twice. let's ask a question

Is the number of heads ≤ 1 ?

If the answer is YES, the possible outcomes are

$$E_{\text{YES}} = \{HT, TH, TT\} \text{ event}$$

If the answer is no, the outcome must be

$$E_{\text{NO}} = \{HH\} \text{ event}$$

However, both E_{YES} and E_{NO} are the subsets of the $\#$ set of possible outcomes when a fair coin is tossed twice. That is, the subset of

$$S = \{HH, HT, TH, TT\}$$

So, an event is defined as the subset of the sample space.

Interestingly, to fully characterize the random experiment, we must know the probability of each of the events.

↳ total number of events is the total possible subsets of the given sample space.

↪ set of possible outcomes once the experiment is conducted

Event space:

The collection of all events is defined as the event space.

If sample space is S , then the event space, denoted as $\mathcal{F}(S)$, is:

$$\mathcal{F}(S) = \{E_1, E_2, E_3, \dots, E_{64}\}$$

for a rolling of a fair die.

□ The nullset ϕ is also event, known as impossible event.

↳ It never occurs in a given experiment.

□ Event space is the power set of the sample space.

Sample space is often represented either as S or Ω .

So, a random experiment is completely characterized by:

$$\{S, \mathcal{F}(S), P(\cdot)\}$$

Sample space event space Probability measure

where, $P(\cdot) : \mathcal{F}(S) \rightarrow [0, 1]$ assigns probability to each of the events in the event space.

↳ with minor modifications this framework is used to describe all of the experiments

So, we need solid understanding of set theory

Basic Set Theory

By definition, a set is a collection of objects. Notationally, curly braces are used to denote a set with comma separated elements.

$$S = \{ _, _, _, _, _ \}$$

The total number of elements in a set is the cardinality of that set. For instance, for the set

$$A = \{ 1, 2, 7, 9, 5 \}$$

the cardinality, denoted as $|A|$, is 5.

☐ Universe, universal set, or space

In any given problem, the set containing all possible elements of interest is called the universe, universal set, or space.

often denoted as S

☐ Set operations

Union of sets: Let's assume that A and B are two sets. Their union, denoted as $A \cup B$, is defined as:

$$A \cup B = \{ x : x \in A \text{ or } x \in B \}$$

It is always assumed that $x \in S$, that is, x is in the space

Intersection of two sets: The intersection of any two sets A and B , denoted as $A \cap B$, is defined as

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

So, $A \cap B$ is the set of elements that belong to both A and B .

$$A = \{1, 7, 5\}, B = \{7\}$$

$$\text{So, } A \cap B = \{7\}$$

complement of a set: Assume that the universe set is S .

with respect to S , the complement of a set A , is denoted as A^c or \bar{A} , is defined as:

$$\bar{A} = \{x : x \notin A\}$$

Empty set: The set that contains no element is known empty set.

Denoted as $\phi, \{\}$

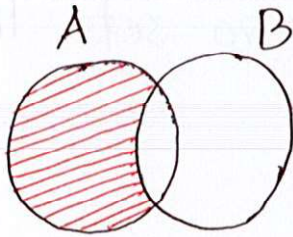
$\{\phi\}$ is not an empty set; instead, it is a singleton set (means, set of single element)

Disjoint set: When two sets have no elements in common $A \cap B = \phi$, and the sets A, B are disjoint.

Set difference:

The set difference between A and B is denoted as $A - B$, and it contains of A that are not in B.

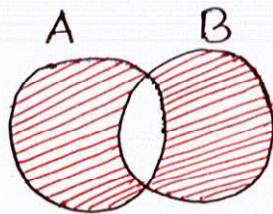
$$A - B = \{x : x \in A \text{ and } x \notin B\} = A \cap \bar{B}$$



Symmetric difference:

For the two sets A and B, the symmetric difference, denoted as $A \Delta B$, is defined as:

$$A \Delta B = \{x : x \in A \text{ or } x \in B, \text{ but not in both}\}$$



Equality of Sets:

Two sets A and B are equal if they contain exactly the same elements.

Two sets A and B are equal if and only if $A \subset B$ and $B \subset A$

Collection of Sets:

are expressed as:

Union of n sets/events $A_1, A_2 \dots A_n$

$$A_1 \cup A_2 \cup \dots \cup A_n$$

shorthand notation

$$\bigcup_{i=1}^n A_i$$

Intersection of n events/sets is expressed as

$$\bigcap_{i=1}^n A_i = A_1 \cap A_2 \dots \dots \cap A_n$$

n sets are mutually exclusive if and only if

$$A_i \cap A_j = \phi, \text{ when } i \neq j$$

when the collection has only two sets, then the two sets are disjoint.

☐ collectively exhaustive

Assume that the universal set S , then the collection of sets $A_1, A_2, \dots \dots A_n$ is collectively exhaustive if and only if

$$A_1 \cup A_2 \dots \cup A_n = S$$

$$\Rightarrow \bigcup_{i=1}^n A_i = S$$

☐ Algebra of Set Theory

$$A \cup B = B \cup A \quad \cup \text{ is commutative}$$

$$A \cap B = B \cap A \quad \cap \text{ is commutative}$$

$$A \cup (B \cap C) = (A \cup B) \cap C \quad \cup \text{ is associative}$$

$$A \cap (B \cup C) = (A \cap B) \cup C \quad \cap \text{ is associative}$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \quad \cap \text{ is distributive over } \cup$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad \cup \text{ is distributive over } \cap$$

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

DeMorgan's Law

$$\bar{\bar{S}} = \phi$$

$$A \cup S = S$$

$$A \cap \phi = \phi$$

$$A \cup \bar{A} = S$$

$$A \cap \bar{A} = \phi$$

Indexed collection of sets

We use index set to form a set of sets.
Assuming that I is the indexed set,

$\{A_i; i \in I\}$ is the indexed collection
of sets

↪ set of sets.

↪ where, there is only one
set A_i for each $i \in I$

Example: Assume $I = \{1, 2, 3, 4\}$. So,

$$\{A_i; i \in I\} = \{A_1, A_2, A_3, A_4\}$$

$$A_1 = [0, 1], A_2 = [0, 2], A_3 = [1, 3], A_4 = [3, 4]$$

Therefore, the indexed collection of sets look as
below:

$$\begin{aligned} & \{A_1, A_2, A_3, A_4\} \\ &= \{[0, 1], [0, 2], [1, 3], [3, 4]\} \end{aligned}$$

☐ Some common index sets:

$$\mathbb{N} = \text{natural numbers} = \{1, 2, 3, \dots\}$$

$$\mathbb{Z}_+ = \text{non-negative integers} = \{0, 1, 2, 3, \dots\}$$

$$\mathbb{Z} = \text{Integers} = \{\dots, -2, -1, 0, 1, 2, 3, \dots\}$$

$$I_n = \{0, 1, 2, \dots, n-1\}$$

$$\mathbb{R} = \text{Real numbers} = (-\infty, +\infty)$$

☐ Some proofs on algebra of sets

✓ $A \cup (A \cap B) = A$ L.H.S = $A \cup (A \cap B) = A + (AB)$

Assuming S as the universal set.

$$= A + AB = AS + AB$$

$$= A(S+B) = AS = A$$

✓ Prove that $(A \cup B) \cap (A \cup C) = A \cup (B \cap C)$

L.H.S = $(A \cup B) \cap (A \cup C) = (A+B)(A+C)$

$$= AA + AB + AC + BC = AS + BC$$

$$= A(A+B+C) + BC = A + BC$$

As, S is the universal set here, we can say that $S = A+B+C$

✓ Another example

Consider the set $\mathbb{Z}_+ \cup \{0\}$, that is set of non-negative real numbers.

We define A_n as:

$$A_n = [0, 1 - \frac{1}{n}]$$

$$= \{x : 0 \leq x \leq 1 - \frac{1}{n}\}$$

where, $n = 1, 2, 3 \dots \dots \dots$

index

So, $A_1 = [0, 1 - \frac{1}{1}] = [0, 1-1] = [0, 0] = [0]$

$A_2 = [0, 1 - \frac{1}{2}] = [0, 0.5] = [0, 0.5]$

Applying the concept of indexed collection of sets

$$\bigcup_{n=1}^{\infty} A_n = A_1 \cup A_2 \cup A_3 \dots \dots \dots$$

$$= [0, \dots, 0.5, \dots 1)$$

open interval, the value is always less than 1.

If we apply DeMorgan laws,

$$\left(\bigcup_{n=1}^{\infty} A_n\right)^c = [0, 1)^c = [1, \infty)$$

Definition:

Suppose, $\{A_i, i \in I\}$ is an indexed family of sets, then the union of the family is

$$\bigcup_{i \in I} A_i = \{x \in S : x \in A_i \text{ for at least one } i \in I\}$$

and the intersection is:

$$\bigcap_{i \in I} A_i = \{x \in S : x \in A_i \text{ for all } i \in I\}$$

↪ sample space

Definition: Collectively Exhaustive

Assume that G is a subset of sample space S . That is, $G \subset S$ and further assume

$$\bigcup_{i \in I} A_i = G$$

Then we can say that the ~~ex~~ family of sets $\{A_i; i \in I\}$ is collectively exhaustive of G .

means that

when we take the unions of the member sets of a family, we get the set G .

□ A family of sets, denoted as $\{A_i; i \in I\}$, is disjoint if all pairs are disjoint. Precisely,

$$A_i \cap A_j = \phi, \forall i, j \in I$$

We use collectively exhaustive and disjoint concept to define a partition.

Partition: A family of sets, denoted as $\{A_i; i \in I\}$ is a partition of the sample space S if it is disjoint and it is collectively exhaustive over the sample space S . That is,

$$\bigcup_{i \in I} A_i = S \quad \text{and} \quad A_i \cap A_j = \emptyset \quad \text{for all } i \neq j$$

✓ The above definition is for a partition over the whole sample space S . However,

the concept can be extended for any subset G of the sample space S .

For instance,

Suppose, $G \subset S$, and $\{A_i; i \in I\}$ is the family of sets. So, $\{A_i; i \in I\}$ is the partition of $G \subset S$ if

$$\underbrace{\bigcup_{i \in I} A_i = G}_{\text{collectively exhaustive}} \quad \text{and} \quad \underbrace{A_i \cap A_j = \emptyset}_{\text{Disjoint}} \quad \text{for all } i \neq j$$

Fact: Assume that $\{A_i; i \in I\}$ is a partition of S , and consider that $G \subset S$. Let's define

$$B_i = A_i \cap G, \quad i \in I$$

then

$\{B_i; i \in I\}$ is a partition of G .

Proof:

Homework

we have to show that

$$\bigcup_{i \in I} B_i = G \quad \text{collectively exhaustive}$$

$$B_i \cap B_j = \emptyset, \quad \text{for } i \neq j \quad \text{Disjoint}$$

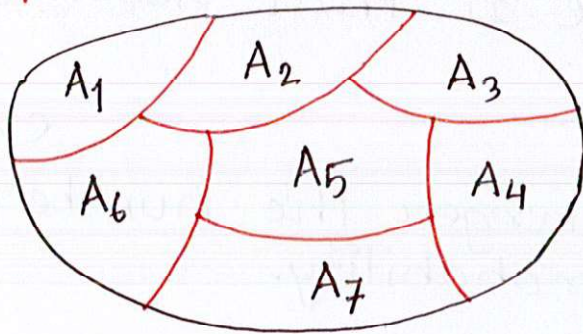
This idea of partition and the related problems are useful for a particular type of decomposition, namely the total probability law.

↳ very important for the proof of Baye's Theorem and related results.

So, with these basic topics covered from Set Theory we now use those to define the —

Probability Space

Example of a partition



$$S = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6 \cup A_7$$

For instance, consider $S = \{1, 2, 3, 4, 5, 6\}$,
and a collection of sets $A_1 = \{1, 2, 3\}$,
 $A_2 = \{4, 5\}$, $A_3 = \{6\}$

As observed, A_1, A_2, A_3 are the partition of S

$$\begin{aligned} A_1 \cup A_2 \cup A_3 &= \{1, 2, 3\} \cup \{4, 5\} \cup \{6\} \\ &= \{1, 2, 3, 4, 5, 6\} \end{aligned}$$

and

$$\begin{array}{l|l} A_1 \cap A_2 = \emptyset & A_1 \cap A_3 = \emptyset \\ A_2 \cap A_3 = \emptyset & \end{array}$$

So, our discussion on set-theory ^{terminologies} directly relates to probability, and can be summarized as follows:

Set theory

Set

Universal set

element

Probability

event

Sample space

outcome

We can state that

Probability is a number between 0 and 1 that describes a set.

event

Also, larger the number, higher the probability.

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Also, larger the number, higher the probability.

Size of the sample space S/Ω

Sample space S of a random experiment, is by definition, is the set of all possible outcomes of the random experiment. For instance,

Finite
sample
space

✓ rolling a fair die has the following sample space

$$S = \{1, 2, 3, 4, 5, 6\}$$

✓ Flipping a fair coin has the sample space as follows

$$S = \{H, T\}$$

✓ If the coin is flipped thrice, the sample space becomes

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

so, as observed here

▣ Different random experiments can have different sample space size.

▣ Size of the sample space S can be finite or infinite.

For instance,

If we flip a coin and the coin is thrown at a different heights, and we are interested to know the speed of the coin when it strikes, the sample space would be

$$S = \mathbb{R}^+$$

whereas,

↳ Uncountable

if we are interested whether its H or T appears in the coin flip, $S = \{H, T\}$ Finite & countable

That is, the same experiment can have different S depending on our desired outcome or event of interest.

Such scenario can be further observed from the classical **Bertrand Paradox**, which states

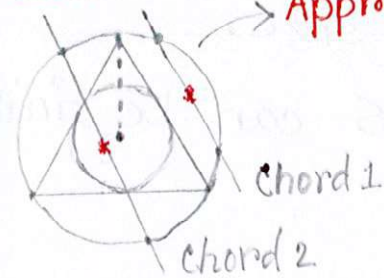
that -

Consider a circle of radius r . Assume that, an equilateral triangle is inscribed in a circle.

Suppose that a chord of the circle is drawn at random. Then,

What is the probability that the chord is longer than the side of the triangle?

Solutions:



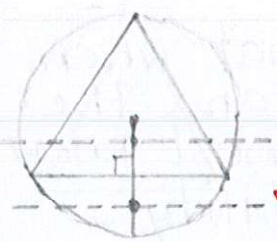
Approach 1: r is the radius
 center of the circle and the centroid of the triangle are same. So, radius of the small circle is $r/2$

As we see, if the mid-point of the randomly drawn chords (Chord 1, Chord 2) are inside the smaller (inscribed) circle, the chord would be greater than the side of the triangle. So, the probability would be $\frac{\pi(r/2)^2}{\pi(r)^2} = \frac{1}{4}$

Approach 2: Select a point on the circle.

equilateral triangle has 60° between two sides. So, when the chord makes an angle $\theta \geq 60^\circ$ & $\theta \leq 120^\circ$, then the chord would be larger than the side of the triangle. So, probability would be

$$\frac{\pi/3}{\pi} = \frac{1}{3}$$



Approach 3: ✓ Consider a radius line, and select a point on the radius.

✓ Draw a chord passing through the point and it is perpendicular to the radius.

✓ Rotate the equilateral triangle and make it parallel to the chord.

✓ The radius and one side of the triangle bisect each other, as they are perpendicular.

Our observation suggests that

when the intersecting point between the radius and the chord is nearer to the center of the circle than the intersecting point between the side of the equilateral triangle and the radius.

So, when the chord intersects the radius at the upper half of the radius. Then the Probability is:

$$\frac{r/2}{r} = \frac{1}{2}$$

For the same question,

we have three logically accurate answers, and these are acceptable answers. But how is it possible?

Answers are different because,

the sample spaces used in these approaches are different.

Approach
1

Sample space
0 to πr^2

2

0 to 2π

3

0 to r

So, the paradox clearly demonstrates that when we talk about the random experiment and the relevant probability of an event, we must know about the exact sample space, probability space.

Also, it is necessary to know if the sample space S is finite or infinite, countable or uncountable] we need set theory again.

☐ Depending on the cardinality of the sample space, they can be classified into the following categories:

Finite sample space

Countably infinite sample space

Uncountable sample space

Thus, it is important to understand the differences between finite and infinite uncountable and countable

☐ Finite vs. Infinite ↗ number of elements

A set is ^{S} finite with cardinality $n \in \mathbb{N}$ if the set $\{0, 1, 2, 3, \dots, n-1\}$ has a bijection to S .

Here, $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ itself is NOT Finite

But, when we take specific number | one-to-one correspondence, say 100, then we should $\frac{1}{100}$

one-to-one
Onto

be able to put the set of our interest, say S , into one-to-one correspondence with set chosen from \mathbb{N} , which is

$\{0, 1, 2, \dots \dots 99\}$ a

finite set.

Simply, A set is finite if it has finite number of elements.

Infinite: A set is infinite if it is not finite.

However, Infinite sets can be classified into two main categories:

- Countable
- Uncountable

countable: An infinite set is countable if its element can be put into one-to-one correspondence with the elements of the set \mathbb{N} .

↳ set of counting numbers.

Uncountable: The infinite set is uncountable if it cannot be put into one-to-one correspondence with the \mathbb{N} .

Example: $\mathbb{R} = (-\infty, +\infty)$

$[0, 1]$ and $(0, 1)$

$[a, b]$, $[a, b)$, $(a, b]$, (a, b) for $a < b$

☐ Show that the odd positive integers is a countable set.

To show that odd positive integers is a countable set, we will show that there is a one-to-one correspondence between this set and the set of positive integers/set of counting numbers.

$$\text{odd positive integers} = \{1, 3, 5, 7, 9, 11, \dots\}$$

$$\mathbb{Z}^+ = \{1, 2, 3, 4, 5, \dots\}$$

If we want to show that a one-to-one $f: \mathbb{Z}^+ \rightarrow \text{odd positive integers}$ exist between the \mathbb{Z}^+ and odd positive integers,

then, all elements of \mathbb{Z}^+ must have unique images in the odd positive integers set, for a $f: \mathbb{Z}^+ \rightarrow \text{odd positive integers}$.

Let's assume,

$$f(n) = 2n - 1$$

↑	1	2	3	4	5	6	7	8	9	10
	1	3	5	7	9	11	13	15	17	19

$$f(n) = 2n - 1$$

↓ so, we obtain one-to-one $f: \mathbb{Z}^+ \rightarrow \text{odd positive integers}$

For onto, assume that t is any odd integer, and our observation says that t is less by 1 from an even integer. Let's say that even integer is $2k$. So, $t = 2k - 1$ for any k integer

$= f(k)$
So, it suggests that all the elements of odd integers must be the image of some elements of \mathbb{N} or \mathbb{Z}^+ . Hence, $f: \mathbb{Z}^+ \rightarrow \text{odd positive integers}$ is onto.

▣ Cardinality and countability

Consider two finite sets:

$$\begin{array}{l} A = \{1, 2, 3, 4, 5, 6\} \\ B = \{2, 4, 6, 8, 10, 12\} \end{array} \left| \begin{array}{l} \text{Cardinality} \\ |A| = 6 \\ |B| = 6 \end{array} \right.$$

Simply, by counting the elements in both sets we can say that $|A| = |B| \equiv$ They are **equicardinal**

Precisely, for finite sets, we can count and compare the ^{total} elements of the sets to decide if they have equal cardinality. However, this approach will not work for sets that are infinite.

For instance, if we want to compare between the set of natural numbers \mathbb{N} and the set of integers \mathbb{Z} **infinite**

the counting approach is not sufficient to compare the two sets \mathbb{N} and \mathbb{Z} .

Mathematician Gregor Cantor suggested that using the concepts of Bijective fnc one can compare two sets, even if the sets are Infinite.

This becomes the most efficient way to assess the size of the set, and is very useful for the sample space.

Gregor Cantor's definition on equicardinality

1. Two sets A and B are equicardinal, denoted as $|A| = |B|$, if there's a bijective (also known as one-to-one correspondence) $f \in C$ between A and B

one-to-one
onto

2. The cardinality of B is greater than or equal to the cardinality of A ($|B| \geq |A|$) if there exists an injective (one-to-one) $f \in C$ from A to B . That is, if

$$f: A \rightarrow B \text{ is an injective } f \in C$$

3. The cardinality of set B is strictly greater than the cardinality of set A if there exists an injective $f \in C$ from A to B , but no bijective $f \in C$ from A to B .

We can use the above definition to define the concept of countability:

Countably infinite: Any set A is countably infinite if A and \mathbb{N} (set of natural numbers) are equicardinal.

A set is countable if it is either finite or countably infinite. So,

we need a bijection $f: A \rightarrow \mathbb{N}$

□ Show that the set of all integers is countable. done later

□ Show that the set of positive rational numbers is countable.

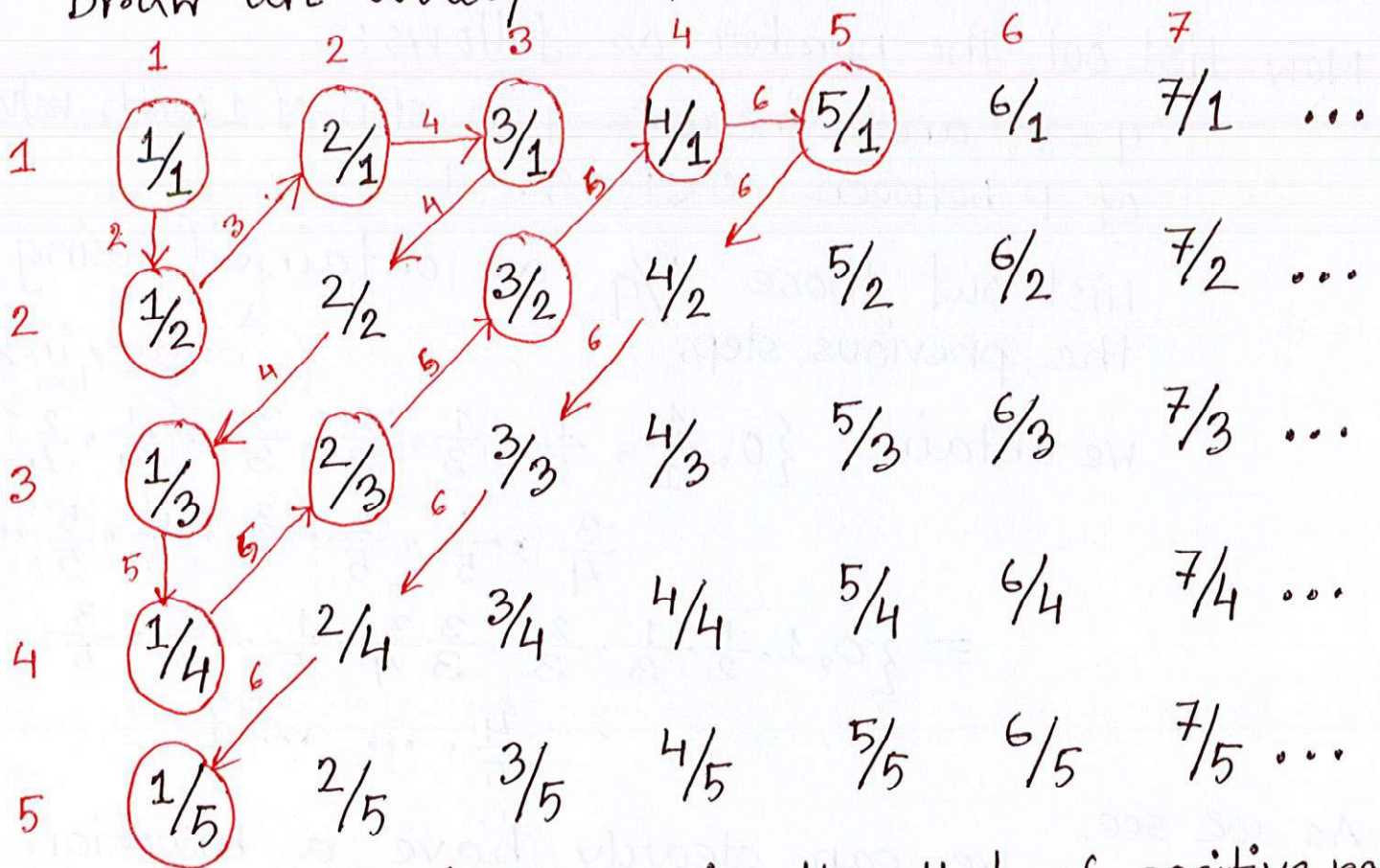
The rational numbers set \mathbb{Q} is defined

$$\mathbb{Q} = \left\{ \frac{p}{q}, \text{ where } p, q \in \mathbb{Z}, q \neq 0 \right\}$$

So, the set of positive rational numbers is defined as: $\mathbb{Q}^+ = \left\{ \frac{p}{q}, \text{ where } p, q \in \mathbb{N} \right\}$

p :
 q : column

Draw an array as follows:



So, the initial terms in the list of positive rational numbers are: $1, \frac{1}{2}, 2, 3, \frac{1}{3}, \frac{1}{4}, \frac{2}{3}, 4, 5, \dots$

positive rational numbers are listed as a sequence. $r_1, r_2, r_3, r_4, r_5, r_6, r_7, \dots, r_n$

So, the set of positive rational numbers is countable.

□ Show that the rationals in $[0, 1]$ is countable

We already have shown that positive rationals set is countable. Here, we consider an interval $[0, 1]$ and we have to show that rationals in $[0, 1]$ is countable.

how to show

we have to show a bijection from $[0, 1]$ to \mathbb{N} .

Again, we create an array similar to previous proof as follows:

Let's assume that p/q is a rational number with $p, q \in \mathbb{Z}^+$ and $q \neq 0$

Now list out the number as follows:

$q=1$, and increase q in step of 1 with value of p between $0 \leq p \leq q$. \perp

List out those p/q as obtained using the previous step.

we obtain $\{0, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{5}{5}, \dots\}$

already in list

$$\equiv \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \dots, \dots\}$$

As we see, we can clearly have a bijection from $[0, 1]$ to \mathbb{N} .

So, it is countable

Theorem: Assume that I is a countable index set, and A_i is countable as well for every $i \in I$. Then, union-

$$\bigcup_{i \in I} A_i \text{ is countable}$$

The above theorem can be used to prove the already proven statement

"The set of positive rational numbers are countable."

We also can prove it for all the rational numbers. That is:

The set of all Rational numbers is countable.

Proof concept: As we have shown ^{for} $[0, 1]$, all the rational numbers between $[0, 1]$ are countable. That is

$$\mathbb{Q} \cap [0, 1] \text{ is countable}$$

So,

$$\mathbb{Q} \cap [1, 2] \text{ is } "$$

$$Q_i = \mathbb{Q} \cap [i, i+1]$$

$$\mathbb{Q} \cap [2, 3] \text{ is } "$$

We can apply,

$$\mathbb{Q} \cap [n, n+1] \text{ is countable}$$

A countable union of countable sets is countable $\forall n \in \mathbb{Z}$

$$\bigcup_{i \in \mathbb{Z}} Q_i = \mathbb{Q}$$

where, each Q_i is countable

How to check one-to-one concept for any $f: A \rightarrow B$?

For any $f: A \rightarrow B$, [check that if $f(a) = f(b)$, then it implies that $a = b$]
it ensures the unique images for each element.

consider, $f(n) = 2n - 1$ | $f(b) = 2b - 1$
so, $f(a) = 2a - 1$ | let, $f(a) = f(b)$
That is, $2a - 1 = 2b - 1$ | $\Rightarrow a = b$
 $\Rightarrow 2a = 2b$

Another example: $f(x) = \frac{x-3}{x+2}$

Assume, a, b are elements in the domain.

$$f(a) = \frac{a-3}{a+2}, \quad f(b) = \frac{b-3}{b+2}$$

let $f(a) = f(b)$, then $\frac{a-3}{a+2} = \frac{b-3}{b+2}$

so, $a = b$
That is, the $f: A \rightarrow B$ is one-to-one

$$\begin{aligned} \Rightarrow (a-3)(b+2) &= (a+2)(b-3) \\ \Rightarrow \cancel{ab} - 3b + 2a - \cancel{6} &= \cancel{ab} + 2b - 3a - \cancel{6} \\ \Rightarrow -3b + 2a &= 2b - 3a \\ \Rightarrow 5a &= 5b \end{aligned}$$

Another example: $f(x) = 1 - x^2$

so, $f(a) = f(b)$
 $\Rightarrow 1 - a^2 = 1 - b^2$
 $\Rightarrow a^2 = b^2$
 $\Rightarrow a = \pm b$

that is,
if $a = 1$,
 $b = \pm 1$

Assume that
Domain \mathbb{Z} or
Co-Domain \mathbb{R}

so, not
one-to-one

□ show that the set of integers is countable

$$\mathbb{Z} = \text{set of integers} = \{0, \pm 1, \pm 2, \pm 3 \dots \dots\}$$

As we know, if we can show that \mathbb{Z} can be put into one-to-one correspondence with \mathbb{N} , then we can say that \mathbb{Z} is countable.

For instance, one can define the bijection between \mathbb{Z} and \mathbb{N} . That is,

$$f: \mathbb{Z} \rightarrow \mathbb{N}$$

One way we can put \mathbb{Z} in one-to-one correspondence with \mathbb{N} is the following:

$$\begin{array}{ccccccc} \dots & -3 & , & -2 & , & -1 & , & 0 & , & 1 & , & 2 & , & 3 & , & \dots \\ & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \\ & 7 & & 5 & & 3 & & 1 & & 2 & & 4 & & 6 & & \end{array}$$

Or, simply as follows:

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \dots & \dots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \\ 0 & +1 & -1 & +2 & -2 & +3 & -3 & +4 & & \end{array}$$