

Statistical Independence

Definition:

Given (S, \mathcal{F}, P) , assume that $A, B \in \mathcal{F}$ then the events A and B are statistically independent if and only if $P(A \cap B) = P(A)P(B)$

In a simpler notion, we can say that
Two events are independent if $P(A \cap B) = P(A)P(B)$
 $\Rightarrow P(AB) = P(A)P(B)$

Properties:

Given that, A and B are statistically independent, then

\bar{A} and B are independent

A and \bar{B} are independent

\bar{A} and \bar{B} are independent

Proof: As A, B are statistically independent

$$P(A \cap B) = P(AB) = P(A)P(B)$$

we have to show that $P(A \cap \bar{B}) = P(A)P(\bar{B})$

Let's write A as the disjunction of two mutually exclusive sets.

$$A = (A \cap B) \cup (A \cap \bar{B})$$

We used $P(A) = 1 - P(\bar{A})$

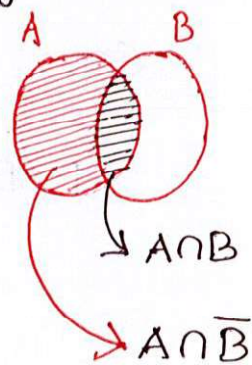
$$\Rightarrow P(A) = P(A \cap B) + P(A \cap \bar{B})$$

form of formula

$$\Rightarrow P(A) = P(A)P(B) + P(A \cap \bar{B})$$

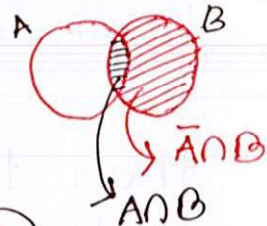
$$\Rightarrow P(A) - P(A)P(B) = P(A \cap \bar{B})$$

$$\Rightarrow P(A) [1 - P(B)] = P(A \cap \bar{B}) = P(A)P(\bar{B})$$



Again, we can write B as a disjunction of two sets:

$$B = (A \cap B) \cup (\bar{A} \cap B)$$



$$\Rightarrow P(B) = P(A \cap B) + P(\bar{A} \cap B)$$

$$\Rightarrow P(B) = P(A)P(B) + P(\bar{A} \cap B)$$

$$\Rightarrow P(B) - P(A)P(B) = P(\bar{A} \cap B)$$

$$\Rightarrow P(B) [1 - P(A)] = P(\bar{A} \cap B)$$

$$\Rightarrow \underline{P(B)P(\bar{A})} = P(\bar{A} \cap B)$$

\hookrightarrow shows that they are statistically independent

□ Consider that we toss a coin twice

Assume, $P(H) = a$, $P(T) = b$ and

$$a + b = P(H) + P(T) = 1$$

Let's consider

$$E_1 = \{H \text{ at first toss}\}$$

$$E_2 = \{H \text{ at second toss}\}$$

Show that the events are statistically independent.

Answer: sample space for two consecutive tosses

$$S = \{HH, HT, TH, TT\}$$

$$\text{So, } E_1 = \{HH, HT\}, \quad E_2 = \{HH, TH\}$$

$$\begin{array}{l|l} P(E_1) = P(HH) + P(HT) & P(E_2) = P(HH) + P(TH) \\ = a^2 + ab & = a^2 + ba \\ = a(a+b) & = a(a+b) \\ = a & = a \end{array}$$

$$\text{Now, } P(E_1 \cap E_2) = P(E_1)P(E_2) = a$$

$$\text{Also, } E_1 \cap E_2 = \{HH\} \text{ and } P(HH) = a^2$$

Independence of three events

The three events A_1, A_2, A_3 are statistically independent

if they are independent in pairs

$$P(A_1 \cap A_2) = P(A_1) P(A_2)$$

$$P(A_2 \cap A_3) = P(A_2) P(A_3)$$

$$P(A_3 \cap A_1) = P(A_3) P(A_1)$$

and

$$P(A_1 \cap A_2 \cap A_3) = P(A_1) P(A_2) P(A_3)$$

Also, we can write

$$\begin{aligned} P(A_1 \cap A_2 \cap A_3) &= P(A_1) P(A_2) P(A_3) \\ &= P(A_1) P(A_2 A_3) \end{aligned}$$

So, any single event is independent of the intersection of the other two.

Q. What about $P(A_1 \cap A_2 \cap \bar{A}_3)$?

$$A_1 A_2 = A_1 A_2 A_3 \cup A_1 A_2 \bar{A}_3$$

$$\Rightarrow P(A_1 A_2) = P(A_1 A_2 A_3) + P(A_1 A_2 \bar{A}_3)$$

$$= P(A_1 A_2) P(A_3) + P(A_1 A_2 \bar{A}_3)$$

$$\Rightarrow P(A_1 A_2 \bar{A}_3) = P(A_1 A_2) - P(A_1 A_2) P(A_3)$$

$$\text{So, replacing } A \text{ by } \bar{A} \text{ results that}$$
$$= P(A_1 A_2) [1 - P(A_3)]$$

$$\text{the events are} \quad = P(A_1 A_2) P(\bar{A}_3)$$

$$\text{independent} \quad = P(A_1) P(A_2) P(\bar{A}_3)$$

Answer

We can generalize our observation on statistically independent events. For instance, let's consider that we have "n" events, and we have been able to define independence of k events for every possible $k < n$.

Then, we can say that events A_1, A_2, \dots, A_n are independent if any $k < n$ of them are independent. So,

$$P(A_1, A_2, \dots, A_n) = P(A_1) P(A_2) \dots P(A_n)$$

Combined Experiments

Consider two experiments

- Rolling a fair die

$$S_1 = \text{sample space} = \{1, 2, 3, 4, 5, 6\}$$

$$P(\text{any face of the die}) = \frac{1}{6}$$

- Flipping a coin

$$S_2 = \{H, T\}, \quad P(H) = P(T) = \frac{1}{2}$$

For instance, we perform both experiments and we want ^{the} probability that we get two on the die and "H" on the coin.

How do we achieve it?

✗

We can combine the experiment and form a new sample space.

Suppose, we have two random experiments
 $(S_1, \mathcal{F}_1, P_1)$ and $(S_2, \mathcal{F}_2, P_2)$

To combine them, we take cartesian product between sample space and form a new sample space

Here, elements of S are ordered pairs

$(a, b) \in S$ with

$a \in S_1$ and $b \in S_2$

$$S = S_1 \times S_2$$

Based on the two experiments of rolling a die and flipping a coin, we can form a combined experiment.

$$\text{So, } S = S_1 \times S_2 = \{(H,1), (H,2), (H,3), (H,4), (H,5), (H,6), (T,1), (T,2), (T,3), (T,4), (T,5), (T,6)\}$$

□ An event in the new combined experiment will be a subset of the sample space

$$S = S_1 \times S_2$$

Now, if $A \subset S_1$ and $B \subset S_2$, then

$C = A \times B$ is a subset (\subset) of S and will be an event in the new event space \mathcal{F} .

For instance, $A = \{H\}$, $B = \{3, 6\}$, we have $A \times B = \{(H,3), (H,6)\}$ which certainly is in the new event space \mathcal{F} , as $\{(H,3), (H,6)\} \subset S$.

However, not all events can be written as the cartesian products.

↳ How will we then form the event space in the combined experiment?

Answer would be, to form the σ -field generated by all cartesian products.

$$\sigma(\{A \times B : \forall A \in \mathcal{F}_1 \text{ and } \forall B \in \mathcal{F}_2\})$$

↳ This is going to be the event space for the combined experiment

Interestingly, the closure properties of σ -field takes care of the events that cannot be written as the cartesian product of the events from \mathcal{F}_1 and \mathcal{F}_2

For instance, $S_1 = \{1, 2, 3, 4, 5, 6\}$
 $S_2 = \{H, T\}$

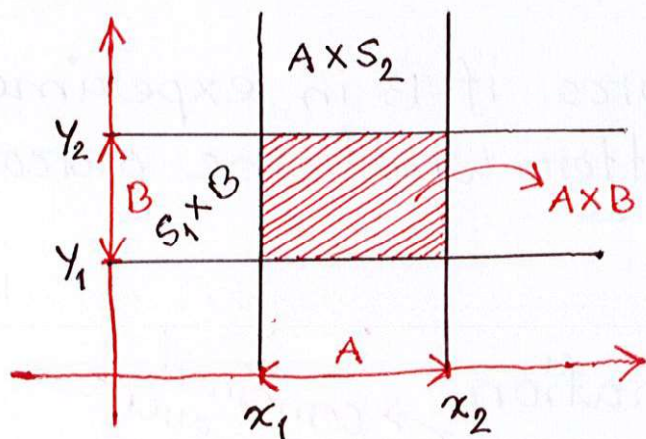
Define an event $C = \{H\} \times \{2, 5\} = \{(H, 2), (H, 5)\}$
 another event $D = \{H, T\} \times \{2\} = \{(H, 2), (T, 2)\}$

Let's define a new event E of the form
 $E = C \cup D = \{(H, 2), (H, 5)\} \cup \{(H, 2), (T, 2)\}$
 $= \{(H, 2), (H, 5), (T, 2)\}$

\square Cartesian Product of two arbitrary sets

Let's assume two sets $A = \{x_1 \leq x \leq x_2\}$

$B = \{y_1 \leq y \leq y_2\}$



As we see, $A \times B$ is the rectangle, and it forms from the two strips.

$A \times S_2$: Vertical strip

$S_1 \times B$: Horizontal strip

Generally, Cartesian product of any two arbitrary sets as a generalized rectangle.

Let's consider S_1 and S_2 are two experiments and we form a new experiment $S = S_1 \times S_2$.

The events from experiment S are all cartesian products of the form: $A \times B$

where, A is an event of S_1
 B is an event of S_2

So, How do we assign probability to a combined set?

Probabilities of the events $A \times S_2$ and $S_1 \times B$ are
also, known as consistency condition $\left[\begin{array}{l} P(A \times S_2) = P_1(A) \\ P(S_1 \times B) = P_2(B) \end{array} \right]$ Assuming that S is a combined experiment

From the rectangular representation we can say that.

Event $A \times S_2$ occurs if A in experiment S_1 occurs no matter what the outcome in S_2 is.

Event $S_1 \times B$ occurs if B in experiment S_2 occurs no matter what the outcome in S_1 is.

With all these information, \rightarrow combined event

How do we define $P(C)$ for other events

$C \in \mathcal{F}$?

\hookrightarrow event space for combined event

A few known information for $P(c)$:

$P(c)$ must satisfy the axioms of probability

$P(c)$ must satisfy the consistency conditions

Generally,

The probabilities of events of the form $A \times B$ and of their unions and intersections can not be expressed in terms of P_1 and P_2 .

↪ We need additional information about S_1 and S_2

Independent Experiments

Often in many experiments, the events $A \times S_2$ and $S_1 \times B$ of the combined experiment S are independent for any A and B . So,

considering the independency,

$$\begin{aligned} P(A \times B) &= P(A \times S_2) \cap P(S_1 \times B) \\ &= P(A \times S_2) P(S_1 \times B) \\ &= P_1(A) P_2(B) \end{aligned}$$

To summarize,

$$A \times S_2 \perp\!\!\!\perp S_1 \times B$$

$\forall A \in \mathcal{F}_1$ Then,
 $\forall B \in \mathcal{F}_2$

$(S_1, \mathcal{F}_1, P_1)$ and $(S_2, \mathcal{F}_2, P_2)$ are independent

So, as we obtain, for two independent experiments $(S_1, \mathcal{F}_1, P_1)$ and $(S_2, \mathcal{F}_2, P_2)$ we have a combined experiment (S, \mathcal{F}, P) with

$$S = S_1 \times S_2$$

$$\mathcal{F} = \sigma(\{A \times B \mid \forall A \in \mathcal{F}_1 \text{ and } \forall B \in \mathcal{F}_2\})$$

$$P(A \times B) = P_1(A) \cdot P_2(B)$$

where, axioms of probability fill in the probabilities of events that can not be written as cartesian products

But those events can be written as a union of disjoint cartesian products.

We can generalize it for n independent experiments:

consider $(S_1, \mathcal{F}_1, P_1), (S_2, \mathcal{F}_2, P_2) \dots (S_n, \mathcal{F}_n, P_n)$ be n random experiments.

Let's form the combined experiment

$$S = S_1 \times S_2 \times S_3 \times \dots \times S_n$$

$$\mathcal{F} = \sigma(\{A_1 \times A_2 \times \dots \times A_n \mid \forall A_1 \in \mathcal{F}_1, \forall A_2 \in \mathcal{F}_2 \dots \forall A_n \in \mathcal{F}_n\})$$

Then,

$$P(A_1 \times A_2 \times \dots \times A_n) = P(A_1) \cdot P(A_2) \cdot P(A_3) \dots P(A_n)$$

and axioms of probability take care of the rest.