Lecture 6-7 Recap

Q. Suppose a fair coin is tossed, what would be the minimum number of attempts we need to try to get a head with 80% certainty?

- The sample space of a fair coin flip is $\{H, T\}$
- The sample space of a sequence of two fair coin flips is all 2^2 possible sequences of outcomes:

$$\{HH, HT, TH, TT\}$$

- The sample space of a sequence of three coin flips is all 2^3 possible sequences of outcomes:

$$\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

$$P(\text{Success after 1 attempt}) = P(\{H\}) = \frac{1}{2} = 0.5$$

$$P(\text{Success after 2 attempts}) = 0.5 + \left(\frac{1}{2} \times \frac{1}{2}\right) = 0.75$$

$$P(\text{Sucess after 3 attempts}) = 0.5 + 0.25 + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$$

$$= 0.875$$

Complement probability: Probability of failure (not getting heads) in a single toss.

For a fair coin, the probability of the failure is the same as the probability of success, both being 0.5.

If we want to find the probability of getting atleast one success in a certain number of attempts, we can use complement probability.

P (at least one success in n attempts) = 1 - P (no success in n attempts)

For one attempt, the probability of no success is 0.5, so the probability of at least one success is

P (at least one success in 1 attempt) = 1 - 0.5 = 0.5

For two attempts, the probability of no success in both attempts is 0.5 × 0.5 = 0.25

For three attempts, the probability of no success in all three attempts is 0.5 × 0.5 × 0.5 = 0.125

• What is the probability of getting one tail of 4 independent coin toss?

Possible combinations where exactly one tail occurs

$$\left.\begin{array}{ll} HHHT & \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} \\ HHTH & \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} \\ HTHH & \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} \\ THHH & \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} \end{array}\right\} \quad 4\times\frac{1}{16} = \frac{1}{4} = 0.25$$

The probability of getting exactly one tail in 4 independent coin tosses is 0.25 or 25%

Event Space $\mathcal{F}(S) \longrightarrow [0, 1]$

Corollaries from axioms: [Basic axioms of probability theory]

(a) $P(A) = 1 - P(A^c)$

(b) $P(A) \leq 1$

(c) $P(\emptyset) = 0$

(d) $P(A=S) = 1$

P(A) represents probability Of event A

Principle of Inclusion - Exclusion

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$\hookrightarrow$ Probability of the union of two events

$$A \cup B = (A - B) \cup (A \cap B) \cup (B - A)$$

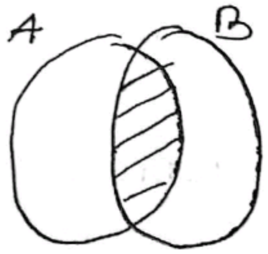
$$P(A \cup B) = P(A - B) + P(A \cap B) + P(B - A)$$

$$= P(A \cap \overline{B}) + P(A \cap B) + P(B \cap \overline{A}) + P(A \cap B)$$

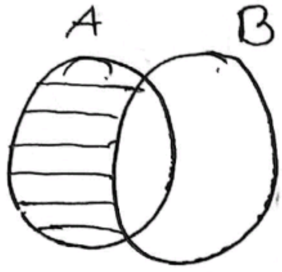
$$- P(A \cap B)$$

$$= P\left((A \cap \overline{B}) \cup (A \cap B)\right) + P\left((B \cap \overline{A}) \cup (A \cap B)\right) - P(A \cap B)$$

$$= P(A) + P(B) - P(A \cap B)$$



$A \cap B$



$A \cap \overline{B}$

$$\overline{B} = U - B$$

Binomial Probability Mass Function

S: Sample Space

$\mathcal{F}(S)$: Event space

$P(\{w\}) : S \rightarrow \mathbb{R}$ such that

(1) $P(\{w\}) \geq 0$

(2) $\sum_{\omega_i \in \mathcal{F}(S)} P(\omega_i) = 1$

Sample space $S: \{H, T\}$

Event space $\mathcal{F}(S): \{\{H\}, \{T\}, \{H, T\}, \emptyset\}$

$$= \{\{H\} \cup \{T\} \cup \{H, T\} \cup \emptyset\}$$

$$= \{H, T\}$$

$$= S$$

$$\sum_{\omega \in S}^{\infty} P(\omega_i) = 1 \quad (\text{Countable / infinite})$$

* Probability Mass Function is used for discrete random variables. Sample space of a discrete random variable is countable.

* Probability Density Function is used for continuous random variables. The sample space of continuous random variable is uncountable

Axioms of Probability

1. $P(A) \geq 0$
   $P(A) \leq 1$ $\quad 0 \leq P(A) \leq 1$

2. $P(S) = 1$ Here, S = sample space

3. $P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$ for $A_1, A_2 \dots$ are disjoint

4. $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$

Candidate Function for PMF

$$d(\omega) : S \rightarrow \mathbb{R} \; [0, 1]$$

$\{\} \rightarrow$ Singleton set

Properties:

1. $\sum_{\omega \in S} d(\omega) = 1$
2. $\omega \in S \quad d(\omega) \geq 0$

Probability $d : \mathcal{F}(S) \rightarrow \mathbb{R}$

Uniform Probability Mass function

Suppose, we have n possible outcomes $\omega_1, \omega_2, \omega_3 \ldots \omega_n$

$$S = \{\omega_1, \omega_2, \omega_3, \ldots, \omega_n\}$$ finite

$$F(S) = 2^{|S|}$$

$| \, |$ ← Cardinality of a set
↓
Number of elements in a set

$$P(A_K) = \sum_{\omega \in A_K} P(\omega) = \sum_{\omega \in A_K} \frac{1}{n}$$

$$A_K = \{\omega_1, \omega_7, \omega_{33}\}$$

$$P(\{\omega_1\}) + P(\{\omega_7\}) + P(\{\omega_{33}\})$$

$$\sum \frac{1}{n} = \frac{1}{n} + \frac{1}{n} + \frac{1}{n} = \frac{3}{n} = \frac{|A_K|}{n} = \frac{|A_K|}{|S|}$$

→ Cardinality of sample space

Binomial PMF

$$P(K) = \binom{n}{k} a^{k} b^{n-k} \qquad a \in [0,1]$$

$$\sum_{\omega \in A} P(\omega) = \sum_{k \in A} P(k) \binom{n}{k} a^{k} (1-a)^{n-k}$$

Is it a valid PMF? Apply two conditions.

$$\sum_{k=0}^{n} \binom{n}{k} a^{k} (1-a)^{n-k} = 1$$

$$(a+b)^{n} = \binom{n}{0} a^{n-0} b^{0} + \binom{n}{1} a^{n-1} b^{1} + \binom{n}{n} a^{n-n} b$$

$$= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k} \qquad [b = 1-a]$$

$$= \sum_{k=0}^{n} a^{n-k} (1-a)^{k}$$

$$= (a + (1-a))^{n}$$

$$= (a + 1 - a)^{n}$$

$$= (1)^{n}$$

Poisson PMF

$S = \{0, 1, 2, 3, \ldots\}$ [Infinite but countable]

$\mathcal{F}(S) = P(S)$

$$K = \frac{e^{-\lambda}\lambda^{K}}{K!} \qquad K = 0, 1, 2, 3, \ldots$$

$$P(A) = \sum_{K \in A} P(K)$$

Valid PMF if $P(K) \geq 0$ and $\sum_{K=0}^{\infty} P(K) = 1$

$$\sum_{K=0}^{\infty} \frac{e^{-\lambda}\lambda^{K}}{K!} = e^{-\lambda}\left(\sum_{K=0}^{\infty} \frac{\lambda^{K}}{K!}\right)$$

$$= e^{-\lambda}e^{\lambda}$$

$$= e^{0}$$

$$= 1$$

Maclaurin Series of e^{x}

$$e^{x} = \sum_{n=0}^{\infty} \frac{x^{n}}{n!}$$

$$= 1 + x + \frac{x^{2}}{2!} + \frac{x^{3}}{3!} + \ldots$$