We start our discussion with

Basic Probability Ideas

and the first topic that we would like to focus on is the SET THEORY

why ?

Precisely, the mathematical basis of probability is the concept of set and set theory.

For any random experiment, we have multiple possible outcomes.

Set concepts provide an easy mechanism to enumerate the underlying possible outcomes of any given random experiment.

For example: consider rolling a fair die

die: has six faces

fair: appearance of each face is equally likely.

So, the possible outcomes are

$$S = \{1, 2, 3, 4, 5, 6\}$$

is a set

Be definition, set is a collection of objects.

could be names, numbers or anything.

Let's define a few events from the random experiment of rolling a fair die.

E_1 = Outcome is odd = $\{1, 3, 5\}$

E_2 = Outcome is divisible by 2 = $\{2, 4, 6\}$

E_3 = Outcome is prime = $\{2, 3, 5\}$

As we see here,

Each of the above events can be represented by a subset of the defined set S.

also, known as sample space

set of all possible outcomes

From this experiment, we can ask

what is the probability that the outcome is odd

or, what is the probability that the outcome is prime

Another experiment: Suppose, a coin is tossed twice

so, the possible outcomes are

	H	T
H	HH	HT
T	TH	TT

$S = \{HH, HT, TH, TT\}$

E_1 = Head (H) appears first = $\{HH, HT,\}$

E_2 = Tail (T) appears first = $\{TH, TT\}$

How many events are possible?

For $S = \{1, 2, 3, 4, 5, 6\}$, distinct subsets would be $2^6 = 64$

Each subset is an event.

consider the coin toss experiment, where the fair coin is tossed twice. Let's ask a question

Is the number of heads ≤ 1 ?

If the answer is YES, the possible outcomes are

$$E_{YES} = \{HT, TH, TT\}$$ event

If the answer is no, the outcome must be

$$E_{NO} = \{HH\}$$ event

However, both E_{YES} and E_{NO} are the subsets of the ~~#~~ set of possible outcomes when a fair coin is tossed twice. That is, the subset of

$$S = \{HH, HT, TH, TT\}$$

So, an event is defined as the subset of the sample space.

Interestingly, to fully characterize the random experiment, we must know the probability of each of the events.

↳ total number of events is the total possible subsets of the given sample space.

set of possible outcomes once the experiment is conducted

Event space:

The collection of all events is defined as the event space.

If sample space is S, then the event space, denoted as $\mathcal{F}(S)$, is:

$$\mathcal{F}(S) = \{E_1, E_2, E_3 \dots \dots E_{64}\}$$

for a rolling of a fair die.

- The nullset ϕ is also event, known as impossible event.
  - It never occurs in a given experiment.
- Event space is the power set of the sample space.

Sample space is often represented either as S or Ω.

So, a random experiment is completely characterized by:

$$\{\underbrace{S}_{\text{Sample space}}, \underbrace{\mathcal{F}(S)}_{\text{event space}}, \underbrace{P(\cdot)}_{\text{Probability measure}}\}$$

where, $P(\cdot) : \mathcal{F}(S) \rightarrow [0, 1]$ assigns probability to each of the events in the event space.

with minor modifications this framework is used to describe all of the experiments

So, we need solid understanding of set theory

Basic Set Theory

By definition, a set is a collection objects. Notationally, curly-braces are used to denote a set with comma separated elements.

$$S = \{\_, \_, \_, \_, \_\}$$

The total number of elements in a set is the cardinality of that set. For instance, for the set

$$A = \{1, 2, 7, 9, 5\}$$

the cardinality, denoted as $|A|$, is 5.

Universe, universal set, or space

In any given problem, the set containing all possible elements of interest is called the universe, universal set, or space.

often denoted as S

Set operations

Union of sets: Let's assume that A and B are two sets. Their union, denoted as $A \cup B$, is defined as:

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

It is always assumed that $x \in S$, that is, x is in the space

Intersection of two sets: The intersection of any two sets A and B, denoted as $A \cap B$, is defined as

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

So, $A \cap B$ is the set of elements that belong to both A and B.

$A = \{1, 7, 5\}$, $B = \{7\}$

So, $A \cap B = \{7\}$

complement of a set: Assume that the universe set is S.

With respect to S, the complent of a set A, ~~is~~ denoted as A^c or $\bar{A}$, is defined as:

$$\bar{A} = \{x : x \notin A\}$$

Empty Set: The set that contains no element is known empty set.

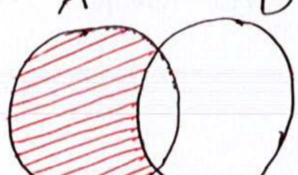
Denoted as ϕ, $\{\ \}$

$\{\phi\}$ is not an empty set; instead, it is a singleton set (means, set of single element)

Disjoint Set: When two sets have no elements in common $A \cap B = \phi$, and the sets A, B are disjoint.

Set difference : The set difference between A and B is denoted as A−B, and it contains of A that are not in B.

$$A-B=\{x: x \in A \text{ and } x \notin B\}$$
$$= A \cap \overline{B}$$



Symmetric difference : For the two sets A and B, the symmetric difference, denoted as $A \Delta B$, is defined as:

$$A \Delta B=\{x: x \in A \text{ or } x \in B, \text{ but not in both}\}$$

Equality of Sets : Two sets A and B are equal if they contain exactly the same elements.

Two sets A and B are equal if and only if $A \subset B$ and $B \subset A$

Collection of Sets : Union of n sets/events $A_1, A_2 \dots A_n$ are expressed as:

$$A_1 \cup A_2 \cup \dots \dots \cup A_n$$

shorthand notation $\bigcup_{i=1}^{n} A_i$

Intersection of n events/sets is expressed as

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \;\dots\;\dots \cap A_n$$

n sets are mutually exclusive if and only if

$$A_i \cap A_j = \phi \;, \quad \text{when } i \neq j$$

when the collection has only two sets, then the two sets are disjoint.

⊞ Collectively exhaustive

Assume that the universal set S, then the collection of sets $A_1, A_2, \dots \dots A_n$ is collectively exhaustive if and only if

$$A_1 \cup A_2 \dots \cup A_n = S$$

$$\Rightarrow \quad \bigcup_{i=1}^{n} A_i = S$$

⊞ Algebra of Set Theory

$A \cup B = B \cup A$ $\cup$ is commutative

$A \cap B = B \cap A$ $\cap$ is commutative

$A \cup (B \cup C) = (A \cup B) \cup C$ $\cup$ is associative

$A \cap (B \cap C) = (A \cap B) \cap C$ $\cap$ is associative

$A \cup S = S$

$A \cap \phi = \phi$

$A \cup \bar{A} = S$

$A \cap \bar{A} = \phi$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $\cap$ is distributive over $\cup$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $\cup$ is distributive over $\cap$

$\overline{A \cap B} = \bar{A} \cup \bar{B}$

$\overline{A \cup B} = \bar{A} \cap \bar{B}$

] DeMorgan's Law

$\bar{S} = \phi$

Indexed collection of sets

We use index set to form a set of sets. Assuming that I is the indexed set.

$\{A_i ;\ i \in I\}$ is the indexed collection of sets

↳ set of sets.

↳ where, there is only one set A_i for each $i \in I$

Example: Assume $I = \{1, 2, 3, 4\}$. So,

$$\{A_i ;\ i \in I\} = \{A_1, A_2, A_3, A_4\}$$

$$A_1 = [0, 1],\quad A_2 = [0, 2],\quad A_3 = [1, 3],\quad A_4 = [3, 4]$$

Therefore, the indexed collection of sets look as below:

$$\{A_1, A_2, A_3, A_4\}$$
$$= \{[0,1], [0,2], [1,3], [3,4]\}$$

Some common index sets:

$\mathbb{N}$ = natural numbers $= \{1, 2, 3 \ldots \ldots\}$

$\mathbb{Z}_+$ = non-negative integers $= \{0, 1, 2, 3, \ldots \ldots\}$

$\mathbb{Z}$ = Integers $= \{\ldots \ldots -2, -1, 0, 1, 2, 3 \ldots \ldots\}$

$I_n = \{0, 1, 2, \ldots \ldots n-1\}$

$\mathbb{R}$ = Real numbers $= (-\infty, +\infty)$