Definition: We recall the concept of random variable X:

Given a random variable X defined on $(S, \mathcal{F}, P)$ that induces a new probability space $(\mathcal{R}, \mathcal{B}(\mathcal{R}), P_X)$, the cumulative distribution function (cdf) of X is defined as

$$
\begin{aligned}
F_X(\alpha) &= P_X((-\infty, \alpha]) = P_X(\{x \mid x \leq \alpha\}) \\
&= P(\{\omega \in S \mid X(\omega) \leq \alpha\}) \\
&= P(X^{-1}((-\infty, \alpha])), \quad \alpha \in \mathbb{R} \\
&= P(\{X \leq \alpha\}), \quad \alpha \in \mathbb{R}
\end{aligned}
$$

where, $\mathcal{R}$ is the range space of X and $\mathcal{R} \subset \mathbb{R}$

Aso, the cdf is the complete probabilistic description of X.

As we already have seen that probabilistic description is a composition between $(\cdot)$ and $P(\cdot)$

$$P_X = P \circ X^{-1}$$

Properties of the cdf:

1. $F_X(+\infty) = 1$ and $F_X(-\infty) = 0$
2. If $x_1 < x_2$, then $F_X(x_1) \leq F_X(x_2)$
3. $P(\{X > x\}) = 1 - F_X(x), \quad \forall x \in \mathbb{R}$
4. If $x_1 < x_2$, then

$$P(\{x_1 < X < x_2\}) = F_X(x_2) - F_X(x_1)$$

5. cdf is right-continuous. That is,

$F_{X}(\cdot)$ $\quad \forall x \in \mathbb{R} \quad \lim_{\varepsilon \downarrow 0} F_{X}(x+\varepsilon) = F_{X}(x)$

6. $P(\{X = x_0\}) = F_{X}(x_0) - F_{X}(x_0^-)$

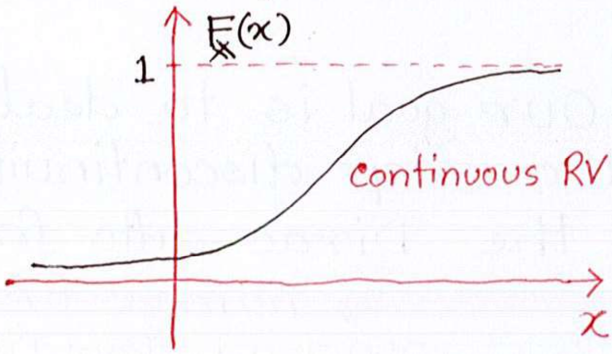
where, $\quad F_{X}(x_0^-) = \lim_{\varepsilon \downarrow 0} F_{X}(x_0 - \varepsilon)$

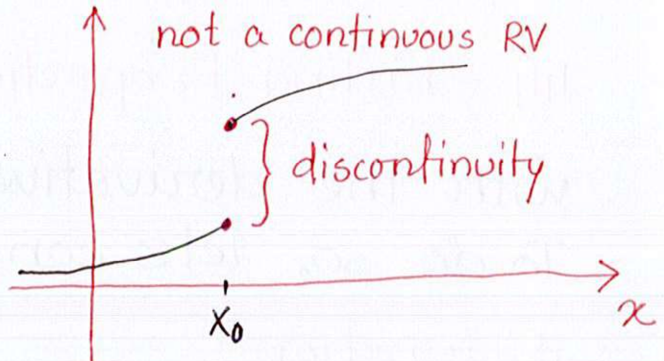
Types of Random Variables

Based on whether the CDF is continuous or not, random variables are subdivided into two subclasses:

1. Continuous random variable
2. Discrete random variable

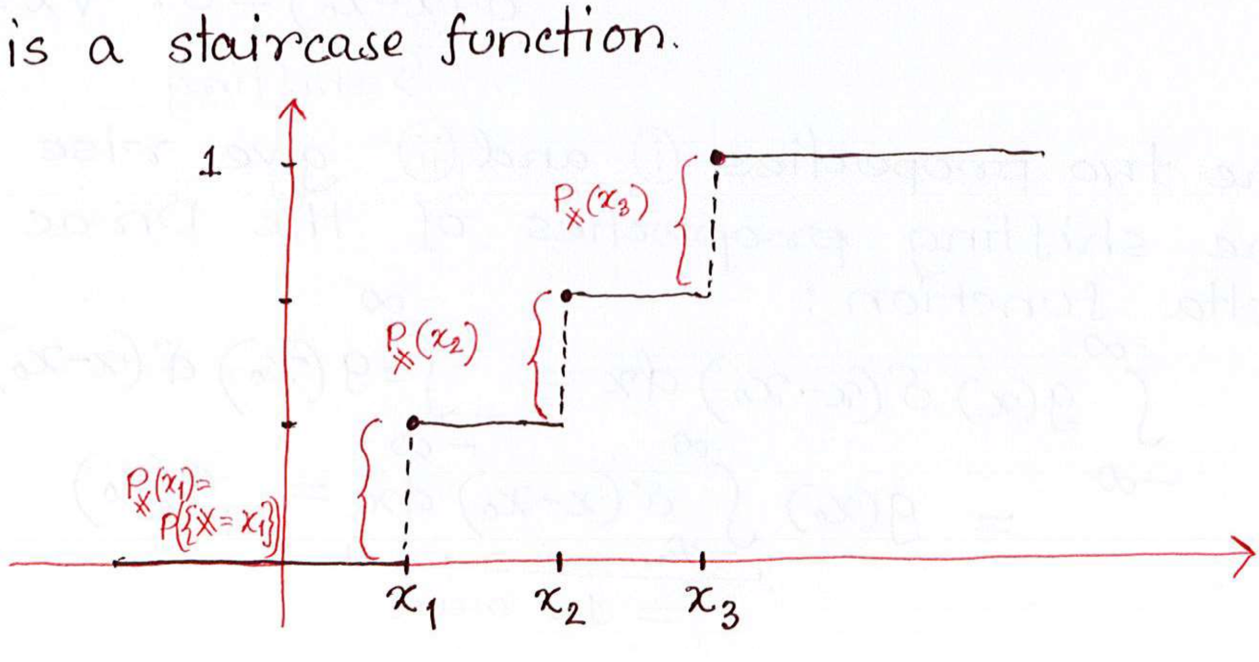
Definition: A random variable is absolutely continuous if $F_X(x)$, that is the cdf, is the continuous function of x. That is, continuous at all points $x \in \mathbb{R}$





Discrete RV: A random variable is discrete if the RV takes on values from a discrete (finite or countable) subset of $\mathbb{R}$.

For a discrete random variable the CDF is a staircase function.



Probability Density Function: The probability density function of a random variable is defined as the derivative of the cdf of the random variable with respect to (w.r.t) x:

$$f_X(x) = \frac{dF_X(x)}{dx}$$

where, $f_X(x) \geq 0 \quad \forall x \in \mathbb{R}$, and

$$\int_{-\infty}^{+\infty} f_X(x)\,dx = F_X(+\infty) - F_X(-\infty) = 1 - 0 = 1$$

Shifting Properties: Our goal is to deal with the derivative of the step discontinuities. To do so, let's consider the Dirac delta fn^c.
↳ named on physicist Paul Dirac

δ-function: Denoted as $\delta(x)$ and defined as:

(i) $\delta(x) = 0, \quad \forall x \neq 0$

(ii) $\int_{-\infty}^{\infty} \delta(x)\,dx = \int_{-\varepsilon}^{\varepsilon} \delta(x)\,dx = 1 \quad \forall \varepsilon > 0$

From (i) we can write

$$\underbrace{\delta(x - x_0)}_{\text{shifting}} = 0, \quad \forall x \neq x_0$$

The two properties (i) and (ii) give rise to the shifting properties of the Dirac delta function:

$$\int_{-\infty}^{\infty} g(x)\,\delta(x - x_0)\,dx = \int_{-\infty}^{\infty} g(x_0)\,\delta(x - x_0)\,dx$$

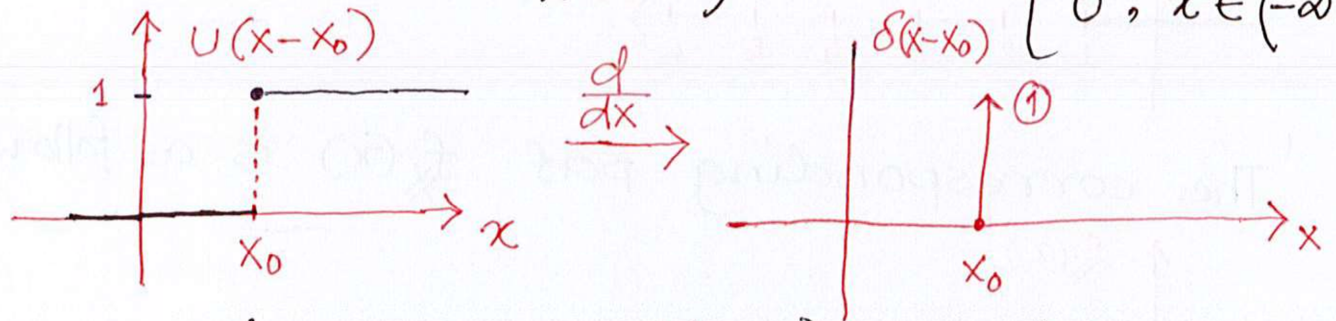
$$= g(x_0) \underbrace{\int_{-\infty}^{\infty} \delta(x - x_0)\,dx}_{= 1,\ \text{area}} = g(x_0)$$

So, we obtain the shifting property as:

$$\int_{-\infty}^{\infty} g(x)\,\delta(x-x_0)\,dx = g(x_0)$$

Example: Suppose, we have

$$U(x-x_0) = 1_{[x_0,\infty)}(x) = \begin{cases} 1, & x \in [x_0, \infty) \\ 0, & x \in (-\infty, x_0) \end{cases}$$



So, $\frac{d}{dx} U(x-x_0) = \delta(x-x_0)$. That is:

$$V(x) = \int_{-\infty}^{x} \delta(r-x_0)\,dr = \begin{cases} 0, & x < x_0 \\ 1, & x > x_0 \\ 1, & x = x_0 \end{cases}$$

Consider a random variable that is the numerical outcomes of rolling a fair die.
each face is equally likely

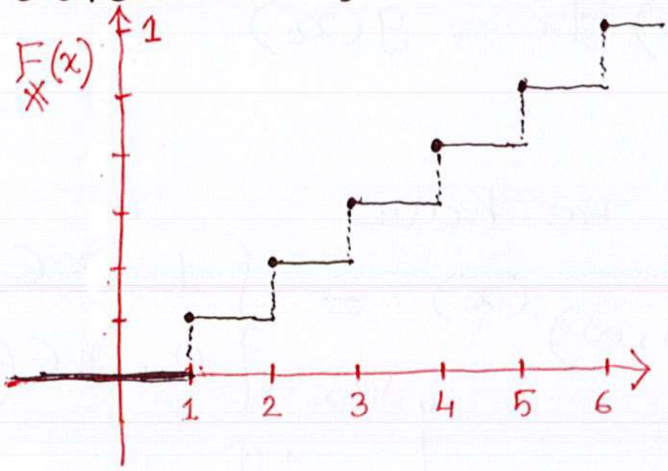
$$F_X(x) = P(\{X \leq x\})$$

$$= \frac{1}{6}\, 1_{[1,\infty)}(x) + \frac{1}{6} \cdot 1_{[2,\infty)}(x) + \frac{1}{6} \cdot 1_{[3,\infty)}(x) + \frac{1}{6} \cdot 1_{[4,\infty)}(x) + \frac{1}{6} \cdot 1_{[5,\infty)}(x) + \frac{1}{6} \cdot 1_{[6,\infty)}(x)$$

$$\Rightarrow f_X(x) = \frac{d}{d} F_X(x)$$

$$= \frac{1}{6}\delta(x-1) + \frac{1}{6}\delta(x-2) + \frac{1}{6}\delta(x-3) + \frac{1}{6}\delta(x-4) + \frac{1}{6}\delta(x-5) + \frac{1}{6}\delta(x-6)$$

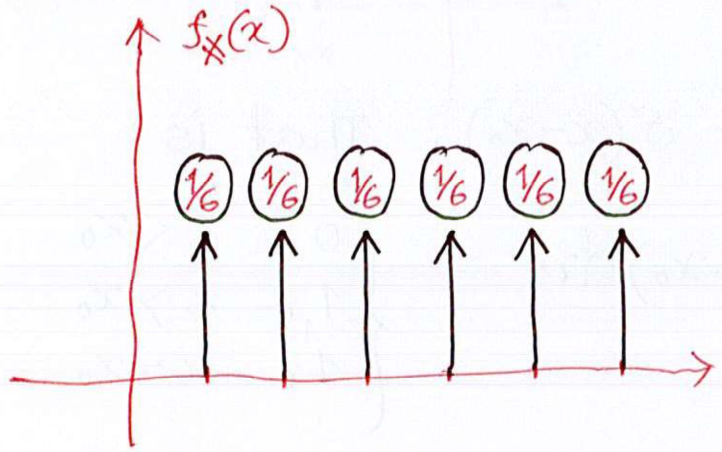
The plots of cdf and pdf of the numerical outcomes of the random variable



For instance,

$$P(\{\mathbb{X} < \quad \}) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

The corresponding pdf $f_{\mathbb{X}}(x)$ is as follows:



Properties of the pdf of a RV:

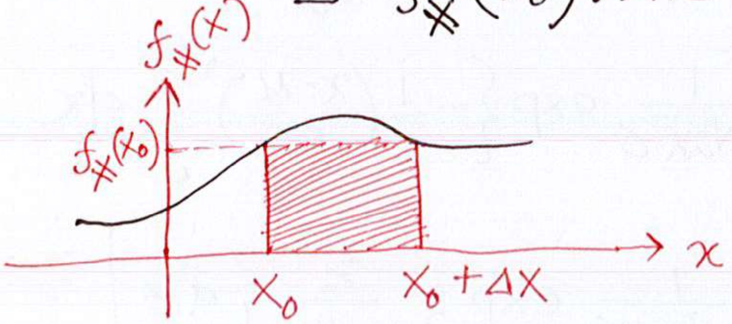
1. $f_{\mathbb{X}}(x) \geq 0, \quad \forall x \in \mathbb{R}$
2. $F_{\mathbb{X}}(x) = \int_{-\infty}^{x} f_{\mathbb{X}}(x)\, dx$
3. $\int_{-\infty}^{\infty} f_{\mathbb{X}}(x)\, dx = 1$
4. $P(\{x_1 \leq \quad \leq x_2\}) = \int_{x_1}^{x_2} f_{\mathbb{X}}(x)\, dx$

$$= F_{\mathbb{X}}(x_2) - F_{\mathbb{X}}(x_1)$$

For a continuous random variable $\mathbb{X}$,

$$P(\{x_0 \leq \mathbb{X} \leq x_0 + \Delta x\}) = \int_{x_0}^{x_0+\Delta x} f_{\mathbb{X}}(x)\, dx$$

$$\cong f_{\mathbb{X}}(x_0) \cdot \Delta x \quad \text{for small } \Delta x$$



As we see, accuracy increases as $\Delta x \to 0$

We can use the fundamental definition of derivative:

$$f_{\mathbb{X}}(x) = \frac{d}{dx} F_{\mathbb{X}}(x)$$

$$= \frac{F_{\mathbb{X}}(x+\Delta x) - F_{\mathbb{X}}(x)}{\Delta x}$$

$$\Rightarrow f_{\mathbb{X}}(x) \cdot \Delta x = F_{\mathbb{X}}(x+\Delta x) - F_{\mathbb{X}}(x)$$

$$= P(\{x < \mathbb{X} < x + \Delta x\})$$

Interestingly, we use pdf or cdf to describe RV and completely ignor the underlying $(S, \mathcal{F}, P)$

However, underlying $(S, \mathcal{F}, P)$ is there, we just don't think about the probality space and move on with the cdf or pdf.

Example: A random variable is known as a Gaussian random variable if it has a pdf of the form:

$$f_{\mathbb{X}}(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{-\frac{(x- \quad)^2}{2\sigma^2}\right\}, \forall x \in \mathbb{R}$$

where, $\mu \in \mathbb{R}$ and $\sigma > 0$

So, using the pdf of Gaussian we can calculate the cdf as follows:

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\, dx = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx$$

$$= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\} dx$$

$$= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{z^2}{2}\right\} dz \qquad z = \frac{x-\mu}{\sigma}$$

$$= \Phi\left(\frac{x-\mu}{\sigma}\right)$$

$$\equiv G\left(\frac{x-\mu}{\sigma}\right)$$

Generally, $\Phi(\cdot)$ cannot be written in "closed form". It is numerically calculated and is widely tabulated.

So, if X is a Gaussian RV with parameters $\mu \in \mathbb{R}$ and $\sigma > 0$, then

$$P(\{a < X < b\}) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

Example: Uniformly distributed RV

A random variable X has a uniform distribution $X \sim U[a, b]$, $a < b$ if

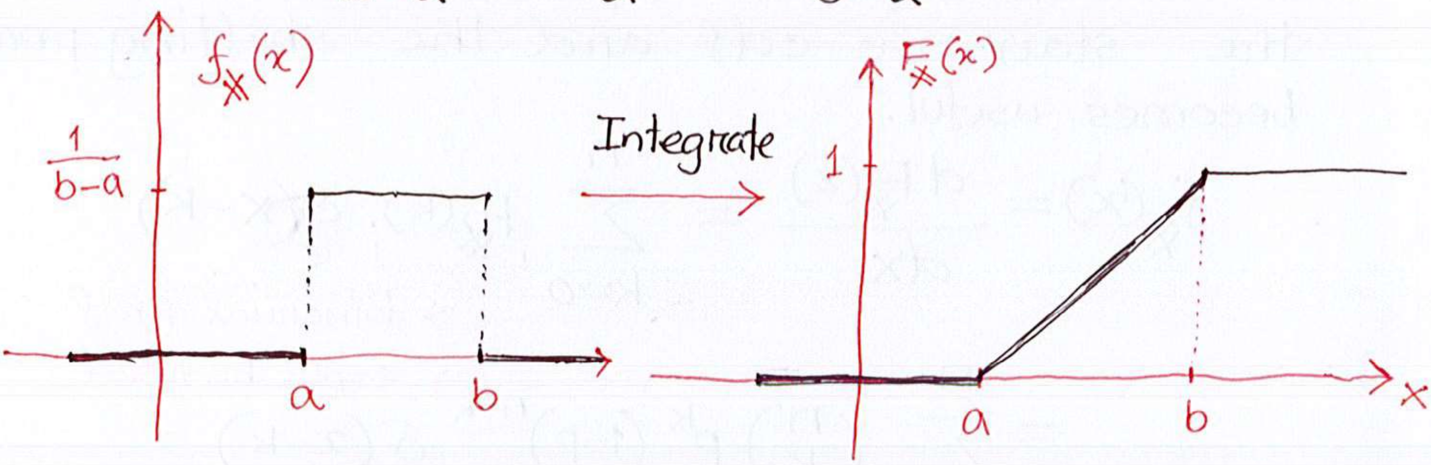
$$f_X(x) = \frac{1}{b-a} \cdot 1_{[a,b]}(x)$$

So, the cdf would be:

$$F_{X}(x) = \int_{-\infty}^{x} f_{X}(r) = \int_{-\infty}^{x} \frac{1}{b-a} \cdot 1_{[a,b]}(r)\, dr$$

$$= \frac{1}{b-a} \int_{-\infty}^{x} dr = \frac{1}{b-a} \int_{+a}^{x} dr$$

$$= \frac{1}{b-a} \left[r \right]_{a}^{x} = \frac{1}{b-a} (x-a)$$



Example: Binomially distributed RV

A binomially distributed RV is a discrete RV. It takes values from the set $\{0, 1, 2, \dots n\} \subset \mathbb{R}$ with pmf:

$$P_{X}(k) = \binom{n}{k} p^{k} (1-p)^{n-k}, \quad \text{where}$$

$k = 0, 1, 2 \dots \dots n$

$p \in [0, 1]$

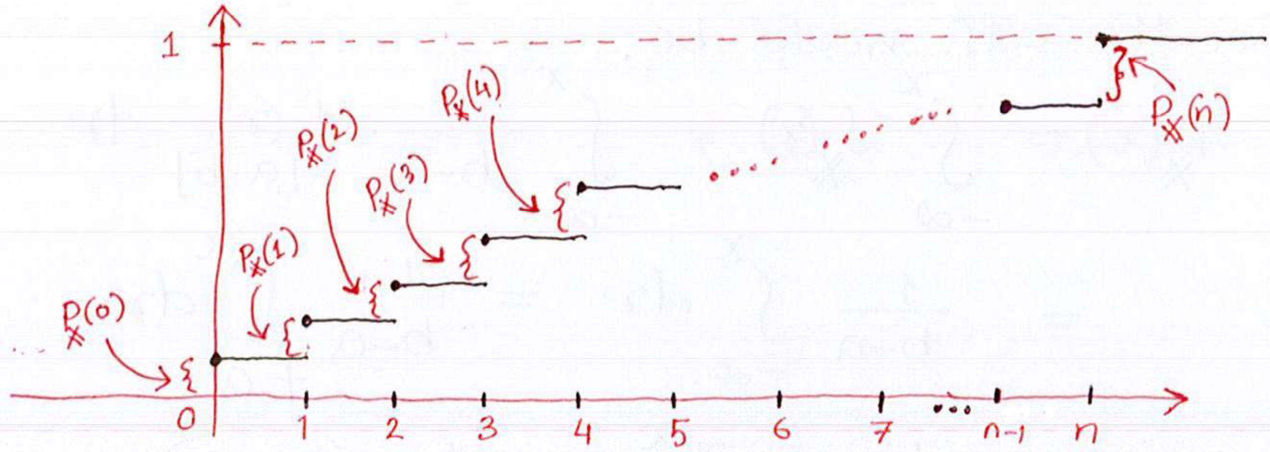
$\hookrightarrow$ probability of occurrence of each favorable outcomes

The cdf of this RV can be calculated as:

$$F_{X}(x) = P(\{X \leq x\})$$

$$= \sum_{k=0}^{m(x)} \binom{n}{k} p^{k} (1-p)^{n-k}$$

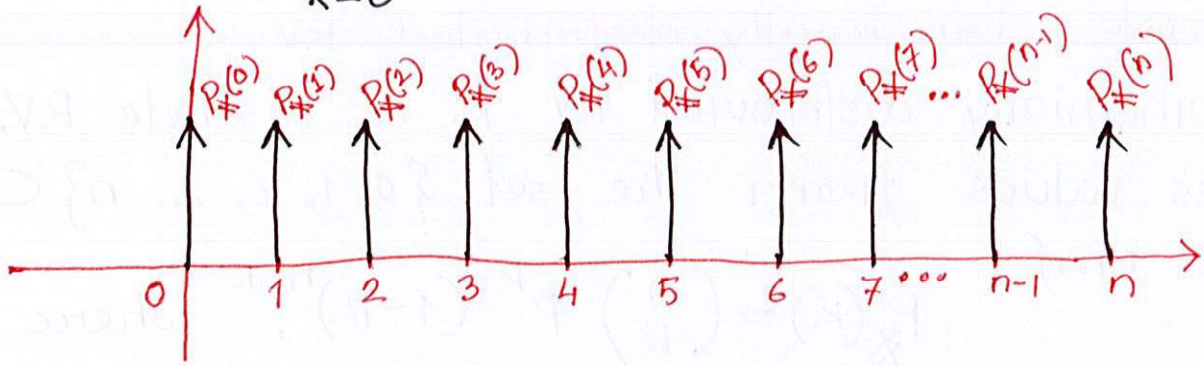
Here, $m(x)$ is an integer such that $m(x) \leq x < m(x)+1$



So, the pdf plot requires the derivative of the staircase cdf and the shifting property becomes useful.

$$f_X(x) = \frac{d F_X(x)}{dx} = \sum_{k=0}^{n} \underbrace{P_X(k)}_{\text{Binomial pmf could be}} \cdot \delta(x-k)$$

$$= \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \delta(x-k)$$

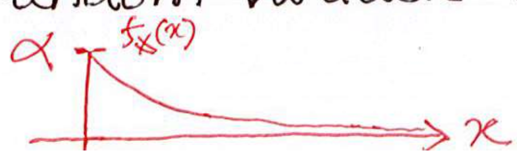


Example : Exponentially Distributed RV

A random variable with a pdf of the form

$$f_X(x) = \alpha e^{-\alpha x} \cdot 1_{[0,\infty)}(x) = \begin{cases} \alpha e^{-\alpha x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

where, $\alpha > 0$, is known as exponential random variable with parameter α.



Let's consider the case $M = \{X \leq a\}$, $a \in \mathbb{R}$

$$F_X(x/M) = P(\{X \leq x\} / \{X \leq a\})$$
$$= \frac{P(\{X \leq x\} \cap \{X \leq a\})}{P(\{X \leq a\})}$$

Now, when $x > a$,

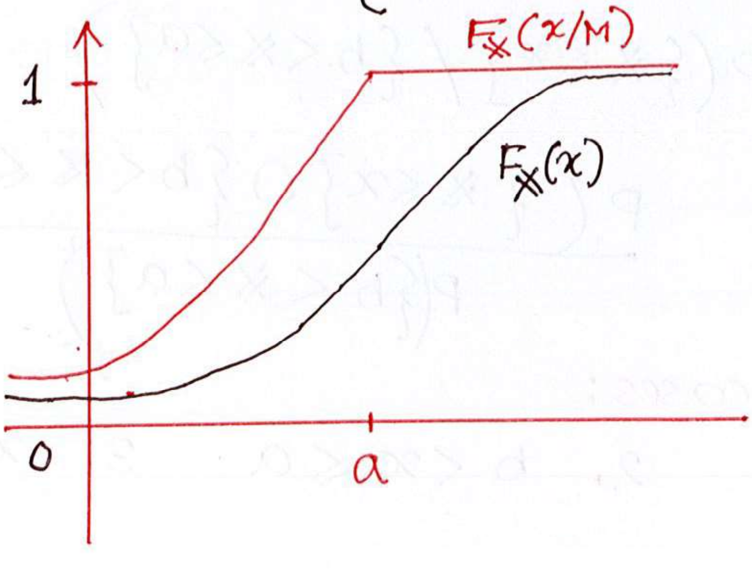
then $P(\{X \leq x\} \cap \{X \leq a\})$
$= P(\{X \leq a\})$

So, $$F_X(x/M) = \frac{P(\{X \leq a\})}{P(\{X \leq a\})} = \frac{F_X(a)}{F_X(a)} = 1$$

when $x \leq a$, then $P(\{X \leq x\} \cap \{X \leq a\})$
$= P(\{X \leq x\})$

So, $$F_X(x/M) = \frac{P(\{X \leq x\})}{P(\{X \leq a\})} = \frac{F_X(x)}{F_X(a)}$$

$$\therefore \quad F_X(x/M) = \begin{cases} F_X(x) & x \leq a \\ 1 & x > a \end{cases}$$



$F_X(a)$ is always smaller than or equal to 1, and hence, red line should be above the black line.

$\Downarrow$

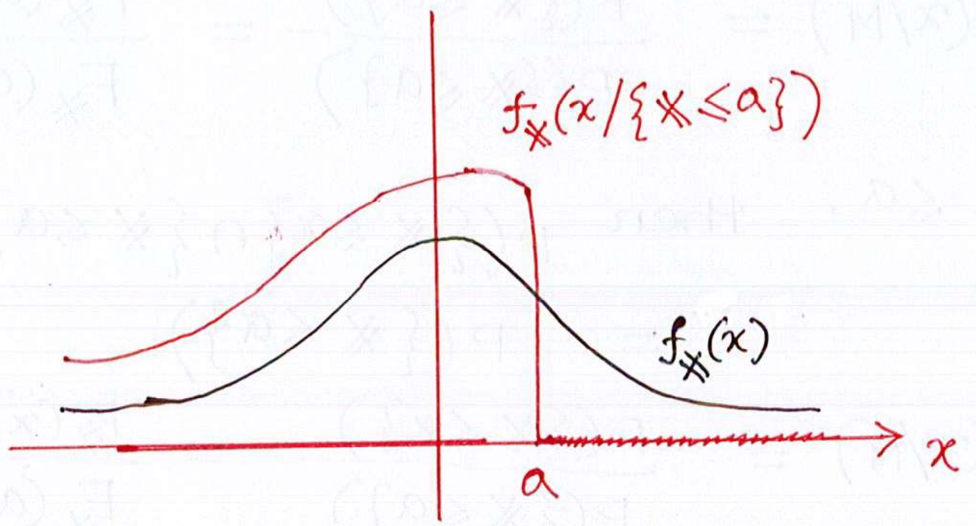
divided by less than one fact

The conditional pdf can be calculated by taking derivatives w.r.t x.

$$f_X(x|\{X \leq a\}) = \frac{dF_X(x|\{X \leq a\})}{dx}$$

$$= \begin{cases} \dfrac{f_X(x)}{F_X(a)}, & x \leq a \\ 0, & x > a \end{cases}$$

A pictorial representation goes as follows:



M coulbe defined as: $M = \{b < X \leq a\}$, $b<a$

so, $F_X(x/M) = F_X(x/\{b < X \leq a\})$

$$= P(\{X \leq x\}/\{b < X \leq a\})$$

$$= \frac{P(\{X \leq x\} \cap \{b < X \leq a\})}{P(\{b < X \leq a\})}$$

Three distinct cases:

1. $x > a$ 2. $b < x \leq a$ 3. $x \leq b$

Conditional Distributions

Given $(S, \mathcal{F}, P)$, assume that X is a random variable defined on the probability space $(S, \mathcal{F}, P)$. Consider that $A, M \in \mathcal{F}$, then

$$P(A/M) = \frac{P(A \cap M)}{P(M)}, \quad P(M) > 0$$

Let's assume, $A = \{X \leq x\} = \{\omega \in S \mid X(\omega) \leq x\}$

Then, $\underline{P(\{X \leq x\} \mid M)} = P(A/M)$

This is the conditional cdf of the RV X condition on event M

Definition: The conditional cdf of the RV X conditioned on $M \in \mathcal{F}$ is

$$F_X(x/M) \triangleq P(\{X \; x\}/M)$$
$$= \frac{P(\{X \leq x\} \cap M)}{P(M)}$$

The definition of $F_X(x/M)$ is similar to that of the $F_X(x)$, except for the fact that the conditional probability measure $P(\cdot/M)$ instead of the probability measure $P(\cdot)$. So,

$$P(\cdot/M) \rightarrow F_X(x/M)$$

Valid probability measure — is a valid cdf

$F_X(x/M)$ has all the properties of a valid cdf.

For instance $P(\{a < X \leq b\}/M) = F_X(b/M) - F_X(a/M)$

Definition: Conditional Probability Density fn^c

The conditional probability density fn^c of random variable X conditioned on $M \in \mathcal{F}$ is:

$$f_X(x/M) \triangleq \frac{d F_X(x/M)}{dx}$$

From the cdf $F_X(x/M)$, we can say that $f(x/M)$ is the valid pdf. That is,

$F_X(x/M)$ is a valid cdf $\longrightarrow$ $f_X(x/M)$ is a valid pdf

As $f_X(x/M)$ is a valid pdf, it also exhibit the necessary properties. For instance

$$P(\{a < X \leq b\}/M) = \int_a^b f_X(x/M)\, dx$$

Comment: In general, we must know the structure of $(S, \mathcal{F}, P)$ and the exact mapping being done through X to determine $F_X(x/M)$ & the conditional pdf $f_X(x/M)$.

Interestingly, Event M could be defined using the RV X. For instance,

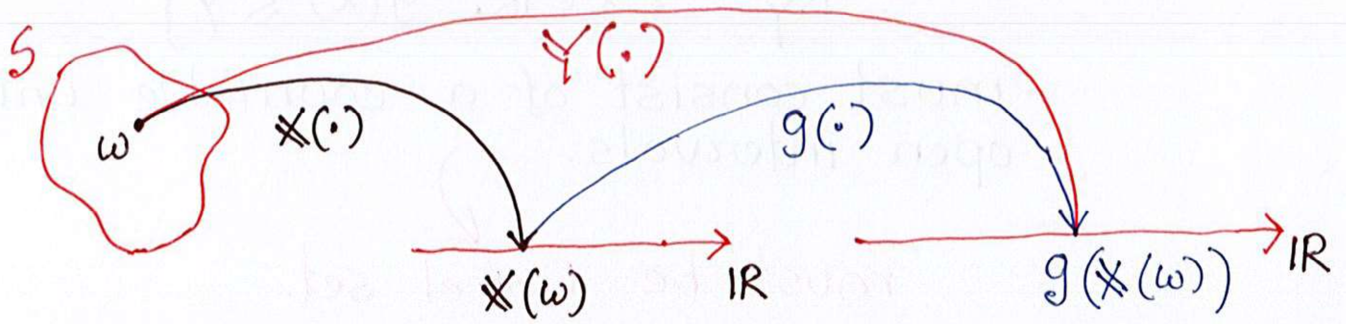
1. $M = \{X \leq a\}$, $a \in \mathbb{R}$
2. $M = \{b < X \leq a\}$, $a, b \in \mathbb{R}$ and $b < a$

Functions of Random Variables

Let's assume that X is a random variable on the probability space $(S, \mathcal{F}, P)$.

We can use this random variable X and define a function Y as follows:

$$Y = g(X), \quad \text{where } g: \mathbb{R} \to \mathbb{R}$$



As we see, the mapping of ω is done to $\mathbb{R}$ twice when we define a function of random variable.

A composite function structure is evident from the definition

$$Y(\omega) = g(X(\cdot)), \text{ that is}$$

$$Y: S \to \mathbb{R}$$

However, the question is

Is $Y(\cdot)$ a random variable ?

To assess it, let's recall the definition:

$Y: S \to \mathbb{R}$ is a random variable if

$$Y^{-1}(A) = \{\omega \in S \mid Y(\omega) \in A\} \in \mathcal{F}, \quad \forall A \in \mathcal{B}(\mathbb{R})$$

where, $\mathcal{F}$ is the event space of $(S, \mathcal{F}, P)$

For $\mathbb{Y} = g(\mathbb{X})$ to be measurable (that is random variable) $g(\cdot)$ must satisfy the following properties

1. The domain of $g(\cdot)$ must contain the range space of $\mathbb{X}$

2. For each $y \in \mathbb{R}$, the set R_y defined as
$$R_Y = \{x \in \mathbb{R};\ g(x) \leq y\}$$
must consist of a countable union of open intervals.

must be Borel set

3. The events $\{g(\mathbb{X}) = \pm\infty\}$ must have probability zero.

So, any function $g(\cdot)$ that satisfies these 3 properties is known as Baire function.

For such $g(\cdot)$, we can say that
$$\mathbb{Y} = g(\mathbb{X})$$
is a valid random variable.

Interestingly,

All functions we typically encounter in engineering applications are Baire functions.

Example: $$Y = g(X) = X^2$$

$\nearrow fn^c$

The transformation considers $g(x) = x^2 = y$
As any specific y is the square of x, y will always be non-negative. So,

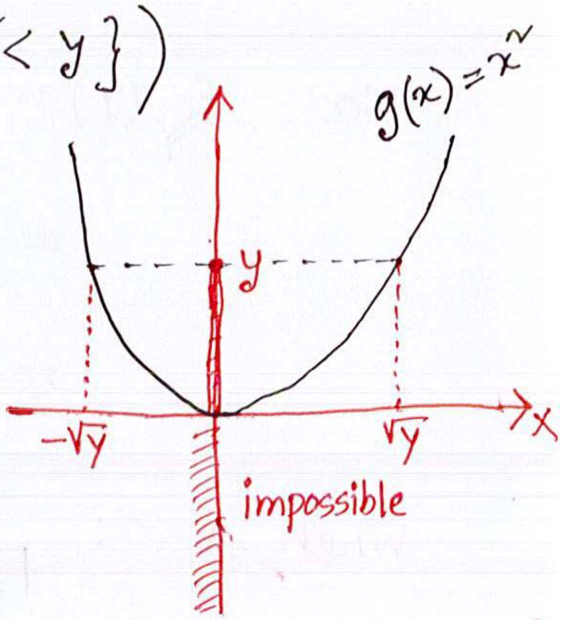
case $y < 0$: $F_Y(y) = 0, \ y < 0$ [negative value of y is impossible]

For the case $y > 0$: Positive y

$$F_Y(y) = P(\{Y \leq y\}) = P(\{X^2 < y\})$$
$$= P(\{-\sqrt{y} \leq X \leq +\sqrt{y}\})$$
$$= P(\{-\sqrt{y} < X \leq +\sqrt{y}\})$$

or, cdf is continuous

for continuous RV case zero probability event can be written without the equal sign



$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

So, the CDF is

$$F_Y(y) = \left[F_X(\sqrt{y}) - F_X(-\sqrt{y})\right] \cdot 1_{(y, \infty)}(y)$$

What about $y = 0$? y becomes zero only at one point x, that is, $x = 0$. Now, if the cdf is continuous at zero, then the probability that $x = 0$ is equal to zero.

$\hookrightarrow$ So, we can ignore the case

$\hookrightarrow (y, \infty)$

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

Examples: $Y = g(X) = aX + b, \quad a, b \in \mathbb{R}$

Find $f_Y(y)$. Two cases $a>0$ and $a<0$

When $a>0$:
$$F_Y(y) = P(\{Y \le y\}) = P(\{aX+b \le y\})$$
$$= P\left(\left\{X \le \frac{y-b}{a}\right\}\right)$$
$$= F_X\left(\frac{y-b}{a}\right)$$

So,
$$f_Y(y) = \frac{d}{dy} F_Y(y)$$
$$= \frac{d}{dy} F_X\left(\frac{y-b}{a}\right) = f_X\left(\frac{y-b}{a}\right) \cdot \frac{d}{dy}\left(\frac{y-b}{a}\right)$$
$$= f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{a} = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

When $a<0$:
$$F_Y(y) = P(\{Y \le y\})$$
$$= P(\{aX+b \le y\}) = P\left(\left\{X \le \frac{y-b}{a}\right\}\right)$$

As $a<0$, it is negative, so,
$$= P\left(\left\{X \ge \frac{y-b}{a}\right\}\right)$$
because $a<0$, the inequality changes
$$= 1 - P\left(\left\{X \le \frac{y-b}{a}\right\}\right)$$
$$= 1 - F_X\left(\frac{y-b}{a}\right)$$

So,
$$f_Y(y) = \frac{d}{dy}\left[1 - F_X\left(\frac{y-b}{a}\right)\right]$$
$$\Rightarrow f_Y(y) = -f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{a} = -\frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

By combining $a<0$, $a>0$, we can write
$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

So, $$f_Y(y) = \frac{dF_Y(y)}{dy}$$

$$\Rightarrow f_Y(y) = \frac{d}{dy}\left[F_X(\sqrt{y}) - F_X(-\sqrt{y})\right] \cdot 1_{(y,\infty)}^{(y)}$$

$$\Rightarrow f_Y(y) = f_X(\sqrt{y})\frac{d}{dy}(\sqrt{y}) - f_X(-\sqrt{y})\frac{d}{dy}(-\sqrt{y})$$

$$= f_X(\sqrt{y})\frac{1}{2\sqrt{y}} - f_X(-\sqrt{y})\frac{-1}{2\sqrt{y}}$$

$$= \frac{1}{2\sqrt{y}}\left[f_X(\sqrt{y}) + f_X(-\sqrt{y})\right]$$

$$= \frac{1}{2\sqrt{y}}\left[f_X(\sqrt{y}) + f_X(-\sqrt{y})\right] \cdot 1_{(0,\infty)}^{(y)}$$