Function of random variables

The direct pdf method:

Suppose $Y = g(X)$, where $g: \mathbb{R} \to \mathbb{R}$ such that $g^{-1}(\cdot)$ exists. That is $y = g(x)$
$\Rightarrow x = g^{-1}(y)$
$\Rightarrow x(y) = g^{-1}(y)$

It is also assumed that

$$\frac{dx}{dy} = \frac{d g^{-1}(y)}{dy} \text{ exists.}$$

Then,
$$f_Y(y) = f_X(g^{-1}(y)) \cdot \left| \frac{d g^{-1}(y)}{dy} \right|$$
where, $x(y) = g^{-1}(y)$

Example: Given that $X \sim U[0,1]$ and let's assume that $Y = g(X) = \sqrt{X}$. So, find $f_Y(y)$.

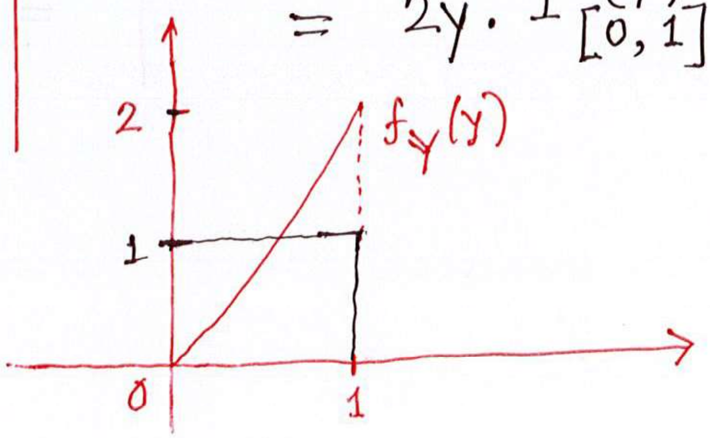
Let's apply direct pdf method

According to the formula:
$$f_Y(y) = f_X(x(y)) \left| \frac{dx(y)}{dy} \right| = f_X(g^{-1}(y)) \left| \frac{d g^{-1}(y)}{dy} \right|$$

Here, $y = g(x) = \sqrt{x}$
$\Rightarrow x = y^2$
$\Rightarrow x(y) = y^2$

So, $\frac{dx(y)}{dy} = 2y$

So,
$$f_Y(y) = |2y| \cdot f(x(y)) = 2y \cdot 1_{[0,1]}(y)$$



Example: Consider X be a Gaussian Random variable with pdf $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

mean μ: 0
variance σ^2: 1

$\mathcal{N}(\mu, \sigma^2)$ is the standard notation for the Gaussian Random Variable with mean μ and variance σ^2.

Consider the linear transformation $Y = aX + b$
Let's find out $f_Y(y)$ using the direct pdf method.

$$f_Y(y) = f_X(x(y)) \left| \frac{dx(y)}{dy} \right|$$

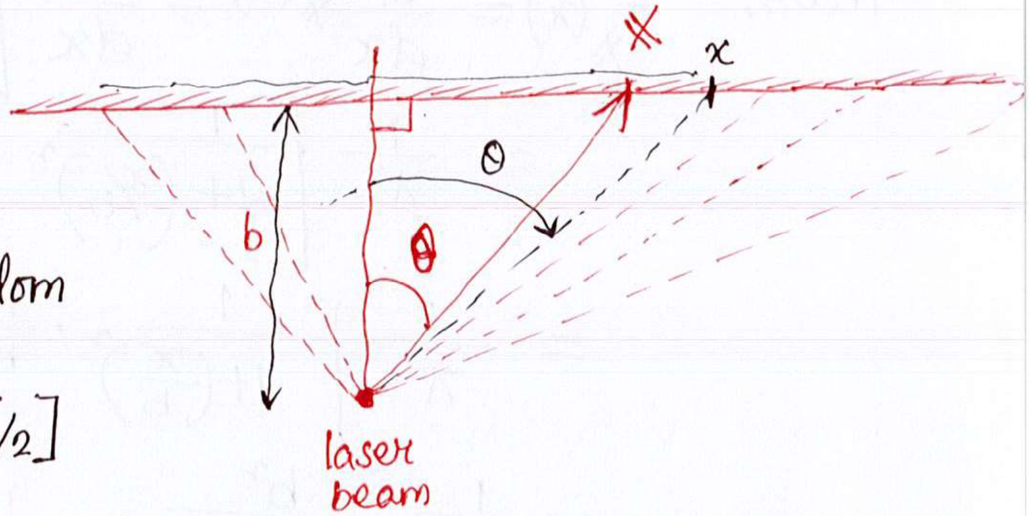
$$y = ax + b$$
$$\Rightarrow x = \frac{y-b}{a}$$
$$\Rightarrow \underbrace{x(y)}_{\text{rewriting considering } x \text{ as a fn}^c \text{ of } y} = \frac{y-b}{a}$$

$$\frac{dx(y)}{dy} = \frac{d}{dy}\left(\frac{y-b}{a}\right) = \frac{1}{a}$$

$$\therefore f = f_X(x(y)) \left| \frac{1}{a} \right|$$
$$= \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-(y-b)^2}{2a^2}\right) \cdot \frac{1}{|a|}$$
$$= \frac{1}{\sqrt{2\pi}\,|a|} \exp\left\{\frac{-(y-b)}{2a^2}\right\}$$

Example: Suppose that a laser beam points at infinitely long wall, and the wall is at a distance b.

Assume that Θ is uniformly distributed random variable on the interval $[-\pi/2, \pi/2]$



Let's say X is the coordinate along the wall where the laser hits. Find the pdf of X

Solution: Let's find out $F_X(x)$ first.

$$\tan\theta = \frac{x}{b}$$
$$\Rightarrow \quad \theta = \tan^{-1}\frac{x}{b}$$
$$\Rightarrow \quad x = b\tan\theta$$
$$\Rightarrow \quad x(\theta) = b\tan\theta$$

So, $P(\overset{\text{random, changes for } \Theta \text{ random}}{X} \leq x)$

$$= P(b\tan\Theta \leq x)$$
$$= P\left(\Theta \leq \tan^{-1}\frac{x}{b}\right)$$
$$= F_\Theta(\tan^{-1} x/b)$$

Now,
$$F_X(x) = P\{X \leq x\}$$
$$= F_\Theta(\tan^{-1} x/b)$$
$$= \frac{\Theta \text{ values generating wall coordinate less than or equal to } x}{\text{total possible } \Theta \text{ values}}$$
$$= \frac{\tan^{-1}(x/b) + \pi/2}{\pi}$$

So, $F_X(x) = \frac{1}{\pi}\left[\tan^{-1}(x/b) + \frac{\pi}{2}\right]$

Thus, $f_X(x) = \frac{d F_X(x)}{dx} = \frac{d}{dx}\left[\frac{1}{\pi}\left(\tan^{-1}x/b + \frac{\pi}{2}\right)\right]$

$$= \frac{1}{\pi}\left[\frac{1}{1+(x/b)^2} \cdot \frac{d}{dx}(x/b) + 0\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{1+\left(\frac{x}{b}\right)^2} \cdot \frac{1}{b}\right] = \frac{1}{\pi b}\left(\frac{1}{\frac{b^2+x^2}{b^2}}\right)$$

$$= \frac{1}{\pi b} \frac{b^2}{b^2+x^2}$$ $\boxed{\text{Answer}}$

Question: Assume that X is a random variable uniformly distributed on $(0, 1)$ and let $Y = g(X)$, where $g(x) = -\lambda \ln(1-x)$ and $\lambda > 0$

Here, $f_X(x) = 1_{[0,1]}^{(x)}$

According to direct method: $f_Y(y) = f_X(x(y))\left|\frac{dx(y)}{dy}\right|$

So, $y = g(x) = -\lambda \ln(1-x)$

$\Rightarrow -\frac{y}{\lambda} = \ln(1-x)$

$\Rightarrow 1-x = e^{-y/\lambda}$

$\Rightarrow x = 1-e^{-y/\lambda}$

$\Rightarrow x(y) = 1-e^{-y/\lambda}$

$\frac{d}{dy}x(y) = \frac{d}{dy}(1-e^{-y/\lambda})$

$= -e^{-y/\lambda} \cdot \lambda)(-1)$

$= e^{-y/\lambda} \cdot 1/\lambda$

So, $f_Y(y) = e^{-y/\lambda} \cdot \frac{1}{\lambda} \cdot 1_{[0,1]}^{(y)}$

Mean, Variance and Expectation

Definition: The mean or expected value of a random variable $\mathbb{X}$ with pdf $f_{\mathbb{X}}(x)$ is defined as:

$$E[\mathbb{X}] \triangleq \int_{-\infty}^{\infty} x\, f_{\mathbb{X}}(x)\, dx \quad \ldots \ldots (1)$$

While the definition as in (1) is for continuous RV, the definition can be extended for discrete RV as well.

For instance, pdf as δ-function

If $P(\{\mathbb{X} = x_K\}) = P_{\mathbb{X}}(x_K) = P_K$ over a discrete index set, then

$$f_{\mathbb{X}}(x) = \sum_K P_{\mathbb{X}}(x_K)\, \delta(x - x_K) = \sum_K P_K\, \delta(x - x_K)$$

and,

$$E[\mathbb{X}] = \int_{-\infty}^{\infty} x\, f_{\mathbb{X}}(x)\, dx = \int_{-\infty}^{\infty} x \left(\sum_K P_K\, \delta(x - x_K) \right) dx$$

$$= \sum_K P_K \int_{-\infty}^{\infty} x\, \delta(x - x_K)\, dx = \sum_K P_K x_K$$

$$= \sum_K P_{\mathbb{X}}(x_K)\, x_K$$ Mean for discrete RV

So, for a discrete RV $\mathbb{X}$, we have

$$E[\mathbb{X}] = \sum_K x_K\, P_{\mathbb{X}}(x_K)$$

Conditional mean of RV

Let's consider $\mathbb{X}$ be a RV on the probability space $(S, \mathcal{F}, P)$ and M is an event in the event space.

Then, the conditional mean of $\mathbb{X}$ conditioned on M is

$$E[\mathbb{X}|M] \triangleq \int_{-\infty}^{\infty} x f_{\mathbb{X}}(x/M)\, dx$$

However, if $\mathbb{X}$ is a discrete RV, the conditional pmf $P_{\mathbb{X}}(x_k/M) = P(\{\mathbb{X}=x_k\}/M)$, then

$$E[\mathbb{X}/M] = \int_{-\infty}^{\infty} x f_{\mathbb{X}}(x/M)\, dx = \int_{-\infty}^{\infty} x \left(\sum_k P_{\mathbb{X}}(x_k/M)\, \delta(x-x_k)\right) dx$$

$$= \sum_k P_{\mathbb{X}}(x_k/M) \cdot \underbrace{\int_{-\infty}^{\infty} x\, \delta(x - x_k)\, dx}_{\text{dirac-}\delta \text{ property}}$$

$$= \sum_k x_k P_{\mathbb{X}}(x_k/M)$$

◫ Mean : exponential

Let X be an exponentially distributed RV with pdf $f_X(x) = \frac{1}{\mu} \exp\{-\frac{x}{\mu}\} 1_{[0,\infty]}(x)$, $\mu > 0$

What is $E[X]$?

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_0^{\infty} \underbrace{x}_{u} \cdot \frac{1}{\mu} \underbrace{e^{-x/\mu}}_{v} dx$$

$$= \int uv\, dx, \text{ where, } u \text{ is a fn}^c \text{ of } x,\ v \text{ is a fn}^c \text{ of } x$$

$$= \frac{1}{\mu}\left[x \int e^{-x/\mu} dx - \int \frac{d}{dx} x \left(\int e^{-x/\mu} dx\right) dx\right]_0^{\infty}$$

$$= \frac{1}{\mu}\left[-x \cdot e^{-x/\mu} \cdot \mu - \int -\mu e^{-x/\mu} dx\right]_0^{\infty}$$

$$= \left[-x e^{-x/\mu} + \int e^{-x/\mu}\right]_0^{\infty}$$

$$= \left[-x e^{-x/\mu} - \mu e^{-x/\mu}\right]_0^{\infty}$$

$$= -\left(-\mu e^{-0/\mu}\right) = \mu e^{-0} = \mu \quad \text{Mean}$$

We can use the same X and conditioned on M.

What would be the expression for $E[X/M]$, where $M = \{X > \mu\}$

According to expectation formula:

$$E[X/M] = E[X/\{X > \mu\}]$$
$$= \int_{-\infty}^{\infty} x f_X(x/\{X > \mu\})\, dx$$

We can calculate $f_X(x/\underbrace{\{X>\mu\}}_{M})$ by evaluating the cdf $F_X(x/M)$.

Then, we can take derivative of $F_X(x/M)$ to obtain the cdf.

We can use the formula as follows:

$$F_X(x/M) = P(\{X \leq x\}/M)$$

$$= \frac{P(\{X \leq x\} \cap M)}{P(M)} = \frac{P(A/\{X \leq x\})F_X(x)}{P(\ )}$$

If, $M = \{X > \mu\}$

$$= \frac{P(\{X>\mu\}/\{X \leq x\})\, F_X(x)}{P(\{X>\mu\})}$$

Question is, How can you calculate $F_X(x)$ from the given $f_X(x)$?

we need this for $F_X(x)$ calculation

$$F_X(x) = \int f_X(x)\, dx$$

$$F_X(x/M) = \int f_X(x/M)\, dx$$

$$P(\{a < X \leq b\}/M) = F_X(b/M) - F_X(a/M)$$

We can write:

$$P(\{a < X \leq b\}/M) = \int_a^b f_X(x/M)\, dx$$

So, $E[X/\{X>\mu\}] = \mu + \mu = 2\mu$. As we obtain $E[X] \neq E[X|\{X>\mu\}]$. Generally,

$$E[X] \neq E[X/M], \quad M \in \mathcal{F}$$

RV as a fn^c of another RV : Expectation

Suppose we have a RV X defined on $(S, \mathcal{F}, P)$ and the pdf is $f_X(x)$.

Consider that a new random variable is defined as: $Y = g(X)$, where $g: \mathbb{R} \to \mathbb{R}$

So, calculate $E[Y] = E[g(X)]$, where $g: \mathbb{R} \to \mathbb{R}$

According to the formula, we can calculate as
$$E[Y] = \int_{-\infty}^{\infty} y \underbrace{f_Y(y)}_{\text{we need evaluation of this.}} dy$$

we can use the below result to calculate $E[Y]$. So,

$$E[g(X)] \triangleq \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

For a discrete random variable, $f_X(x) = \sum_k P_X(x_k)\, \delta(x - x_k)$

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) \left(\sum_k P_X(x_k)\, \delta(x - x_k) \right) dx$$

$$= \sum_k P_X(x_k) \int_{-\infty}^{\infty} g(x)\, \delta(x - x_k)\, dx = \sum_k g(x_k)\, P_X(x_k)$$

So, let's calculate the conditional mean $E[X/M]$ for a random variable with pdf

$$f_X(x) = \frac{1}{\mu} \exp\left\{-\frac{x}{\mu}\right\} \cdot 1_{[0,\infty)}(x)$$

$$E[X/M] = \int x f_X(x/M)\, dx$$

$$= \int x\, f_X(x/\{X > \mu\})\, dx$$

For $M = \{X > \mu\}$

We can show that

$$f_X(x/\{X>\mu\}) = \frac{\frac{1}{\mu}\exp\left\{-\frac{x}{\mu}\right\} \cdot 1_{(\mu,\infty)}(x)}{e^{-\mu/\mu}}$$

$$= \frac{1}{\mu} \exp\left\{-\frac{(x-\mu)}{\mu}\right\} 1_{(\mu,\infty)}(x)$$

$$\therefore E[X|\{X>\mu\}] = \int_{-\infty}^{\infty} x\, f_X(x|\{X>\mu\})\, dx$$

$$= \int_{\mu}^{\infty} x \cdot \frac{1}{\mu} \exp\left\{\frac{-(x-\mu)}{\mu}\right\} dx$$

Consider $r = x - \mu$

$\Rightarrow x = r + \mu$

$\Rightarrow dx = dr$

So, we obtain,

$$= \int_0^{\infty} (r+\mu) \frac{1}{\mu} \exp\left\{\frac{-r}{\mu}\right\} dr$$

$$= \int_0^{\infty} r \cdot \frac{1}{\mu} e^{-\left\{\frac{r}{\mu}\right\}} dr + \int_0^{\infty} \mu \cdot \frac{1}{\mu} e^{-\left\{\frac{r}{\mu}\right\}} \cdot dr$$

$$= \underbrace{\frac{1}{\mu}\int_0^{\infty} r \cdot e^{-\left\{\frac{r}{\mu}\right\}} dr}_{\mu} + \int_0^{\infty} e^{-\frac{r}{\mu}} dr$$

From $E[X]$ derivation

$$\left[-\mu e^{-x/\mu}\right]_0^{\infty} = +\mu \cdot e^0 = \mu \cdot 1 = \mu$$

⊞ Variance: Given a random variable $\mathbb{X}$, the variance of $\mathbb{X}$, denoted as $Var(\mathbb{X})$, is as follows

$$Var(\mathbb{X}) \triangleq E\left[(\mathbb{X}-\bar{\mathbb{X}})^2\right] = \int_{-\infty}^{\infty} (x-\bar{\mathbb{X}})^2 f_{\mathbb{X}}(x)\, dx$$

Here, $\bar{\mathbb{X}} = E[\mathbb{X}]$

often $\sigma_{\mathbb{X}}^2$ is used as the symbol for $Var(\mathbb{X})$

$$\sigma_{\mathbb{X}}^2 = Var(\mathbb{X})$$

Positive square root of the variance of $\mathbb{X}$ is called the standard deviation of $\mathbb{X}$.

$$\text{Std. Dev}(\mathbb{X}) = \sqrt{\sigma_{\mathbb{X}}^2} = \sigma_{\mathbb{X}} = \sqrt{Var(\mathbb{X})}$$

⊞ $Var(\mathbb{X}) = E\left[(\mathbb{X}-\bar{\mathbb{X}})^2\right] = E\left[\mathbb{X}^2 - 2\mathbb{X}\bar{\mathbb{X}} + \bar{\mathbb{X}}^2\right]$

$$= E[\mathbb{X}^2] - E[2\mathbb{X}\bar{\mathbb{X}}] + E[\bar{\mathbb{X}}^2]$$

$$= E[\mathbb{X}^2] - 2\bar{\mathbb{X}}\, E[\mathbb{X}] + E[\bar{\mathbb{X}}^2]$$

$$= E[\mathbb{X}^2] - 2\bar{\mathbb{X}}\cdot\bar{\mathbb{X}} + \bar{\mathbb{X}}^2$$

$$= E[\mathbb{X}^2] - \bar{\mathbb{X}}^2 = E[\mathbb{X}^2] - \left(E[\mathbb{X}]\right)^2$$

So, $Var(\mathbb{X}) = E[\mathbb{X}^2] - \left(E[\mathbb{X}]\right)^2$

Linearity of expectation

Consider that $g_1(X)$ and $g_2(X)$ are the functions of RV X, and α, β are the constants with $\alpha, \beta \in \mathbb{R}$

Then, $$E[\alpha g_1(X) + \beta g_2(X)] = \alpha E[g_1(X)] + g_2(X)\beta$$

Example: Consider a Gaussian RV X with pdf

$\mu \in \mathbb{R}$ and $\sigma > 0$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$$

Find $E[X]$ and $Var(X)$

$$E[X] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} \cdot x \, dx$$

Let $x - \mu = r$

$\Rightarrow x = \mu + r$

$\Rightarrow dx = d\mu + dr$

$\Rightarrow dx = dr$

$$= \int_{-\infty}^{\infty} (\mu + r) \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dr$$

$$= \int_{-\infty}^{\infty} (\mu + r) \exp\left\{-\frac{r^2}{2\sigma^2}\right\} dr$$

$$= \underbrace{\int_{-\infty}^{\infty} r \exp\left\{-\frac{r^2}{2\sigma^2}\right\} d}_{\text{expand it Part A}} + \int_{-\infty}^{\infty} \mu \exp\left\{-\frac{r^2}{2\sigma^2}\right\} dr$$

Part A: $$\frac{1}{\sqrt{2\pi}\sigma}\left[\int_{-\infty}^{0} r \exp\left\{-\frac{r^2}{2\sigma^2}\right\} dr + \int_{0}^{\infty} r \exp\left\{-\frac{r^2}{2\sigma^2}\right\} dr\right]$$

$$= \frac{1}{\sqrt{2\pi}\sigma}\left[-\int_{0}^{-\infty} r \exp\left\{-\frac{r^2}{2\sigma^2}\right\} dr + \int_{0}^{\infty} r \exp\left\{-\frac{r^2}{2\sigma^2}\right\} dr\right]$$

We can rewrite by doing a change of variable.

So, $\frac{1}{\sqrt{2\pi}\,\sigma}\int_0^{-\infty}(-r)\exp\left\{-\frac{r^2}{2\sigma^2}\right\}dr$

Consider $(-r) = r$

So, $dr = -dr$

When $-r = 0$
$\Rightarrow r = 0$

$-r = -\infty$
$\Rightarrow r = \infty$

$$= \frac{1}{\sqrt{2\pi}\,\sigma}\int_0^{-\infty}(-r)\exp\left\{-\frac{(-r)^2}{2\sigma^2}\right\}dr$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma}\int_0^{\infty} r\exp\left\{-\frac{r^2}{2\sigma^2}\right\}(-dr)$$

after the change of variable process

limits are changed

$$= \frac{1}{\sqrt{2\pi}\,\sigma}\cdot(-1)\cdot\int_0^{\infty} r\exp\left\{-\frac{r^2}{2\sigma^2}\right\}dr$$

$$= -\frac{1}{\sqrt{2\pi}\,\sigma}\int_0^{\infty} r\exp\left\{-\frac{r^2}{2\sigma^2}\right\}dr$$

By plugging in these values, we obtain

$$\text{Part A} = \frac{1}{\sqrt{2\pi}\,\sigma}\left[\int_0^{\infty} -r\exp\left\{-\frac{r^2}{2\sigma^2}\right\}dr + \int_0^{\infty} r\exp\left\{-\frac{r^2}{2\sigma^2}\right\}dr\right]$$

$$= 0$$

That is,

$$E[X] = \int_{-\infty}^{\infty}\mu\cdot\frac{1}{\sqrt{2\pi}\,\sigma}\exp\left\{-\frac{r^2}{2\sigma^2}\right\}dr$$

We can do change of variable by assuming

$$\frac{r^2}{2\sigma^2} = z$$

So, $z = \frac{x}{\sqrt{2}\sigma}$ $\Rightarrow x^2 = 2z\sigma^2$

$\Rightarrow dz = \frac{1}{\sqrt{2}\sigma} dx$

$\Rightarrow dz \cdot \sqrt{2}\sigma = dx$ $\Rightarrow \sqrt{2}\sigma\, dz = dx$

Replacing the variable x-term by z, we obtain

$$E[X] = \int_{-\infty}^{\infty} \mu \cdot \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\left(\frac{x}{\sqrt{2}\sigma}\right)^2\right\} dx$$

$$= \int_{-\infty}^{\infty} \mu \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-z^2} \cdot dz.$$

$$= \int_{-\infty}^{\infty} \mu \cdot \frac{1}{\sqrt{\pi}} e^{-z^2} dz$$

$$= \mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-z^2} dz = \mu \cdot \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-z^2} dz$$

$$= \mu \cdot 1 = \mu$$

We know that $\frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-z^2} dz = \lim_{z \to \infty} \int_0^{z} \frac{2}{\sqrt{\pi}} e^{-z^2}$

$= 1.$