田 Example : Poisson Random Variable (RV)

Let's consider that X is a Poisson distributed random variable with pmf

$$P_X(k) = P_k = P(\{X=k\}) = \frac{e^{-\mu}\mu^k}{k!}$$

where, $k = 0, 1, 2, \ldots \ldots$ and $\mu > 0$

Find $E[X]$, $Var(X)$

Mean :

$$E[X] = \sum_{k=0}^{\infty} k\, P_X(k) = \sum_{k=0}^{\infty} k.\, \frac{e^{-\mu}.\, \mu^k}{k!} = \sum_{k=1}^{\infty} k.\frac{e^{-\mu}.\mu^k}{k!}$$

$$= \sum_{k=1}^{\infty} \frac{e^{-\mu}.\, \mu^k}{(k-1)!}$$

Let, $m = k-1$
$\Rightarrow m+1 = k$
When $k = 1$, $m = 0$
$k = \infty$, $m = \infty$

$$= \sum_{m=0}^{\infty} \frac{e^{-\mu}.\, \mu^{m+1}}{(m+1-1)!}$$

$$= \sum_{m=0}^{\infty} \frac{e^{-\mu}.\, \mu^m.\, \mu}{m!} = e^{-\mu}.\mu \underbrace{\sum_{m=0}^{\infty} \frac{\mu^m}{m!}}_{\text{Def}^n \text{ of } e^x}$$

$$= e^{-\mu}.\mu.\, e^{\mu}$$

$$= \mu$$

mean of X

Poisson

Variance:

$$Var(X) = E[X^2] - (E[X])^2$$

$$= E[X^2] - \mu^2$$

$$E[X^2] = \sum_{k=0}^{\infty} k^2 P_X(k) = \sum_{k=0}^{\infty} k^2\, \frac{e^{-\mu}.\, \mu^k}{k!}$$

$$= \sum_{k=0}^{\infty} k.\, \frac{e^{-\mu}.\, \mu^k}{(k-1)!}$$

$$= \sum_{k=1}^{\infty} k \cdot \frac{e^{-\mu}\cdot\mu^{k}}{(k-1)!} = e^{-\mu}\sum_{k=1}^{\infty}\frac{\mu^{k}}{(k-1)!}$$

Change of variable
$k-1=m$
$\Rightarrow m+1=k$
$k=1,\ m=0$
$k=\infty,\ m=\infty$

$$= e^{-\mu}\sum_{m=0}^{\infty}\frac{\mu^{m+1}}{m!} = e^{-\mu}\sum_{m=0}^{\infty}\frac{\mu^{m}\cdot\mu}{m!}(m+1)$$

$$= e^{-\mu}\cdot\mu\cdot\sum_{m=0}^{\infty} m\frac{\mu^{m}}{m!} + e^{-\mu}\cdot\mu\cdot\underbrace{\sum_{m=0}^{\infty}\frac{\mu^{m}}{m!}}_{\text{Formula}}$$

$$= e^{-\mu}\cdot\mu\sum_{m=0}^{\infty}\frac{m\,\mu^{m}}{m!} + e^{-\mu}\cdot\mu\cdot e^{\mu}$$

$$= \underbrace{\mu\sum_{m=0}^{\infty} m\cdot\frac{e^{-\mu}\cdot\mu^{m}}{m!}}_{\text{mean of Poisson}} + \mu\cdot 1 = \mu^2+\mu$$

So, $\operatorname{Var}(X) = E[X^2] - (E[X])^2$

$$= \mu^2 + \mu - (\mu)^2$$

$$= \mu$$

Interestingly, mean and variance of a poisson distributed random variable are the same.

The Characteristic Function

The characteristic function, if attributed to the random variable, means the Fourier Transform of the underlying probability distribution of the random variable.

Let X be a random variable on $(S, \mathcal{F}, P)$, the characteristic function of X is:

$$\Phi_X(\omega) \triangleq E\left[e^{i\omega X}\right], \quad \omega \in \mathbb{R}$$

$$= \underbrace{\int_{-\infty}^{\infty} e^{i\omega x} f_X(x)\, dx}$$

Complex as we have i here. $i^2 = -1$

That is $\Phi_X(\omega) : \mathbb{R} \to \mathbb{C}$

maps real-valued ω to complex values.

We know that $e^{i\theta} = \cos\theta + i\sin\theta$ [Euler's Formula]

so, $E\left[e^{i\omega X}\right] = E\left[\cos\omega X + i\sin\omega X\right]$

$$= E\left[\cos\omega X\right] + i\,E\left[\sin\omega X\right]$$

linearity of expectation

Q. For characteristic function $\Phi(\omega)$, does the integral exist for all "ω"?

How do we know that the integral does not blow up.

We can calculate the modulus of $\Phi_X(\omega)$, also can be termed as $|\Phi_X(\omega)|$

$$= \left| \int_{-\infty}^{\infty} f_X(x)\, e^{i\omega x} \right|$$

$$\leq \int_{-\infty}^{\infty} | f_X(x)\, e^{i\omega x} |\, dx$$

expected value of an integral is always less than or equal to the integral of the expected value of the integrand

We can write

$$\int_{-\infty}^{\infty} | f_X(x)\, e^{i\omega x} |\, dx$$

$$= \int_{-\infty}^{\infty} \underbrace{|f_X(x)|}_{\text{pdf o } X \text{ is never negative}}\, |e^{i\omega x}|\, dx$$

$$= \int_{-\infty}^{\infty} f_X(x) \cdot 1\, dx = \int_{-\infty}^{\infty} f_X(x) = 1$$

So, $|\Phi_X(\omega)| \leq \underbrace{\Phi_X(0)}_{=1}$

That is, $\Phi_X(\omega)$ is well defined for any $f_X(x)$

Also,

If we know the characteristic function, we can extract the pdf from it using inverse fourier transform.

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_X(\ )\, e^{-i\omega x}\, d\omega$$

complete probabilistic description

$\Phi_X(\omega)$ is a complete probability description

Example: If X is an exponentially distributed RV with mean μ, find $\Phi_X(\omega)$.

Ans: $\Phi_X(\omega) = E\left[e^{i\omega X}\right] = \int_{-\infty}^{\infty} e^{i\omega x} f_X(x)\, dx$

$$= \int_{-\infty}^{\infty} e^{i\omega x} \cdot \frac{1}{\mu} \cdot e^{-x/\mu} \cdot 1_{[0,\infty)}(x)\, dx$$

$$= \int_{0}^{\infty} e^{i\omega x} \cdot \frac{1}{\mu} e^{-x/\mu}\, dx = \frac{1}{\mu}\int_0^{\infty} e^{\left(i\omega - \frac{1}{\mu}\right)x} dx$$

we can follow the change of variable approach

$\left(i\omega - \frac{1}{\mu}\right) = z$

when $x = 0$
$z = 0$ $\Rightarrow$ $dx = \frac{dz}{\left(i\omega - \frac{1}{\mu}\right)}$

when $x = \infty, z = \infty$

$$= \frac{1}{\mu}\int_0^{\infty} e^{z} \cdot dz \cdot \left(\frac{1}{i\omega - 1/\mu}\right) = \frac{1}{\mu} \times \frac{\mu}{}$$

$$= \frac{1}{1 - i\omega\mu} \qquad \boxed{\text{Ans.}}$$

Properties: If X is a random variable and it is used to define another RV Y as $Y = aX + b$, $a, b \in \mathbb{R}$

Then, $\Phi_Y(\omega) = e^{i\omega b}\, \Phi_X(a\omega)$

Proof: $\Phi_Y(\omega) = E\left[e^{i\omega}\right] = E\left[e^{i\omega(aX+b)}\right]$

$$= E\left[e^{i\omega a X} \cdot e^{i\omega b}\right] = e^{i\omega b} E\left[e^{i\omega a X}\right]$$

$$= e^{i\omega b}\, \Phi_X(a\omega)$$

Moment Generating function : mgf

The moment generating function of any RV is defined as: $\Phi_X(s) \triangleq E\left[e^{sX}\right]$, where $s \in \mathbb{R}$ or, $s \in \mathbb{C}$

complex values

Simpler cases consider s to be real-valued.

If we consider $s \in \mathbb{C}$, then $\Phi_X(s)$ is a bilateral Laplace transformation.

$$\Phi_X(s) \triangleq E\left[e^{sX}\right] = \int_{-\infty}^{\infty} f_X(x)\, e^{sx}\, dx$$

That is: if we let $s \in \mathbb{C}$. $\Phi_X(\omega) = \Phi_X(i\omega)$

so, $\Phi_X(\ ) : \mathbb{C} \to \mathbb{C}$

if $s \in \mathbb{R}$, then $\Phi_X(\cdot) : \mathbb{R} \to \mathbb{R}$

With moment generating function, we can calculate n^{th} moment of X.

Moment Theorem: Given a RV X with mgf $\Phi_X(s)$, the n-th moment of X is given by:

$$E\left[X^n\right] = \left.\frac{d^n \Phi_X(s)}{ds^n}\right|_{s=0} = \Phi_X^{(n)}(0)$$

derivative notation

Proof: Let's differentiate $\Phi_X(s)$ w.r.t s for n-times:

$$\frac{d}{ds}\Phi_X^1(s) = \frac{d}{ds} E\left[e^{sX}\right] = E\left[X e^{sX}\right]$$

So, the second derivative can be obtained as:

$$\Phi_X^{(2)}(\ ) = \frac{d}{ds}\Phi_X^{(1)}(s) = \frac{d}{ds} E[e^{sX}]$$

$$= E\left[\frac{d}{ds} X e^{sX}\right] = E[X X e^{sX}]$$

$$= E[X^2 e^{sX}]$$

$$\vdots$$

$$\Phi_X^{(n)}(s) = E[X^n e^{sX}]$$

That is, $\Phi_X^{(n)}(0) = E[X^n \cdot e^{0X}] = E[X^n]$

Example: Consider that X is an exponentially distributed RV with mean μ. Find its variance.

Answer ↓

We already have obtained

$$\Phi_X(\omega) = \frac{1}{1-i\omega\mu} = (1-i\omega\mu)^{-1}$$

$$Var(X) = E[X^2] - (E[X])^2 = E[X^2] - \mu^2$$

So, $E[X^2] = \frac{d^2}{ds^2}\Phi_X(s)\Big|_{s=0}$

$$= \frac{d^2}{ds^2}\left[(1-s\mu)^{-1}\right]\Big|_{s=0}$$

$$= \frac{d}{ds}\frac{d}{ds}\left[(1-s\mu)^{-1}\right]\Big|_{s=0} = \frac{d}{ds}\left[-(1-s\mu)^{-2}(-\mu)\right]\Big|_{s=0}$$

$$= \left[-2\mu(1-s\mu)^{-3}(-\mu)\right]\Big|_{s=0} = 2\mu^2(1-s\mu)^{-3}\Big|_{s=0} = 2\mu^2$$

so, $\operatorname{var}(X) = E[X^2] - \mu^2 = 2\mu^2 - \mu^2 = \mu^2$

We can also use characteristic function

$$E[X^n] = \frac{1}{i^n} \frac{d^n \Phi_X(\omega)}{d\omega^n}\Big|_{\omega=0}$$

If we identify $\Phi_X(\omega)$ with $\Phi_X(i\omega)$

$$E[X^n] = \frac{d^n \Phi_X(i\omega)}{d(i\omega)^n}\Big|_{i\omega=0}$$

Example: For the exponential case, the char. function is

$$\Phi_X(\omega) = (1-i\omega\mu)^{-1}$$

$$E[X] = \frac{d}{d(i\omega)} \Phi(\omega)\Big|_{i\omega=0} = -(1-i\omega\mu)^{-2}(-\mu)\Big|_{i\omega=0}$$

$$= \frac{\mu}{(1-i\omega\mu)^2}\Big|_{i\omega=0} = \frac{\mu}{(1-0)^2} = \mu$$

Answer

Fact: Let's consider X is a Gaussian RV, the characteristic function (C.F) for X would be:

$$\Phi_X(\omega) = e^{i\omega\mu} \cdot e^{-\frac{1}{2}\omega^2\sigma^2}$$
$$= e^{i\omega\mu} \cdot e^{\frac{1}{2}(i^2)\cdot\omega^2\cdot\sigma^2}$$
$$= e^{i\omega\mu} \cdot e^{\frac{1}{2}(i\omega)^2\sigma^2}$$
$$= \Phi_X(i\omega) \equiv \Phi_X(\omega)$$

We also can write the moment generating function

$$\Phi_X(s) = e^{s\mu} \cdot e^{\frac{1}{2}s^2\sigma^2}$$

where, $s = i\omega$ gives back the characteristic function

We can use moment theorem

$$E[X] = \frac{d}{ds}\Phi_X(s)\Big|_{s=0}$$
$$= \frac{d}{ds} e^{s\mu} \cdot e^{\frac{1}{2}s^2\sigma^2}\Big|_{s=0}$$
$$= \mu e^{s\mu} \cdot e^{\frac{1}{2}s^2\sigma^2} + e^{s\mu} \cdot e^{\frac{1}{2}s^2\sigma^2} \cdot \frac{1}{2} \times 2s \times \sigma^2\Big|_{s=0}$$
$$= \mu e^{s\mu} \cdot e^{\frac{1}{2}s^2\sigma^2} + e^{s\mu} \cdot \sigma^2 \cdot e^{\frac{1}{2}s^2\sigma^2} \cdot s\Big|_{s=0}$$
$$= \mu \cdot 1 \cdot 1 + 1 \cdot \sigma^2 \cdot 0 \cdot 1$$
$$= \mu$$

Answer

$$Var(X) = E[X^2] - (E[X])^2$$
$$= E[X^2] - \mu^2$$

$$E[X^2] = \Phi_X^{(2)}(0)$$

We can use moment generating function

$$E[X^2] = \frac{d}{ds}\frac{d}{ds}\Phi_X(s)\Big|_{s=0} = \frac{d}{ds}\left(\mu e^{\mu s}\cdot e^{\frac{1}{2}s^2\sigma^2} + e^{\mu s}\sigma^2 s\, e^{\frac{1}{2}\sigma^2 s^2}\right)$$

$$= \Big|\mu\cdot\mu e^{\mu s}\cdot e^{\frac{1}{2}s^2\sigma^2} + \mu\cdot e^{\mu s}\cdot e^{\frac{1}{2}s^2\sigma^2}\cdot s\sigma^2 + e^{\mu s}\cdot\sigma^2\cdot s\, e^{\frac{1}{2}\sigma^2 s^2}\cdot s\sigma^2 + e^{\frac{1}{2}\sigma^2 s^2}\cdot\left(e^{\mu s}\cdot\sigma^2 + s\sigma^2\mu\cdot e^{\mu s}\right)\Big|_{s=0}$$

$$= \mu^2 + 1(\sigma^2 + 0) = \mu^2 + \sigma^2$$

$$\therefore \quad Var(X) = \mu^2 + \sigma^2 - (\mu)^2 = \sigma^2$$ Ans