Indexed collection of sets

We use index set to form a set of sets. Assuming that I is the indexed set.

$\{A_i ; i \in I\}$ is the indexed collection of sets

set of sets.

where, there is only one set A_i for each $i \in I$

Example: Assume $I = \{1, 2, 3, 4\}$. So,

$$\{A_i ; i \in I\} = \{A_1, A_2, A_3, A_4\}$$

$A_1 = [0, 1]$, $A_2 = [0, 2]$, $A_3 = [1, 3]$, $A_4 = [3, 4]$

Therefore, the indexed collection of sets look as below:

$$\{A_1, A_2, A_3, A_4\}$$

$$= \{[0, 1], [0, 2], [1, 3], [3, 4]\}$$

Some common index sets:

$\mathbb{N}$ = natural numbers $= \{1, 2, 3 \dots \dots\}$

$\mathbb{Z}_+$ = non-negative integers $= \{0, 1, 2, 3, \dots \dots\}$

$\mathbb{Z}$ = Integers $= \{\dots \dots -2, -1, 0, 1, 2, 3 \dots \dots\}$

$I_n = \{0, 1, 2, \dots \dots n-1\}$

$\mathbb{R}$ = Real numbers $= (-\infty, +\infty)$

Some proofs on algebra of sets

✓ $A \cup (A \cap B) = A$

Assuming S as the universal set.

$$\begin{aligned} L.H.S &= A \cup (A \cap B) = A + (AB) \\ &= A + AB = AS + AB \\ &= A(S+B) = AS = A \end{aligned}$$

✓ Prove that $(A \cup B) \cap (A \cup C) = A \cup (B \cap C)$

$$\begin{aligned} L.H.S &= (A \cup B) \cap (A \cup C) = (A+B)(A+C) \\ &= AA + AB + AC + BC \\ &= A(A+B+C) + BC \end{aligned}$$

$$\begin{aligned} &= AS + BC \\ &= A + BC \end{aligned}$$

As, S is the universal set here, we can say that $S = A+B+C$

✓ Another example

Consider the set $\mathbb{Z}_+ \cup \{0\}$, that is set of non-negative realnumbers.

We define A_n as:

$$A_n = \left[0, 1 - \frac{1}{n}\right] = \left\{x : 0 \leqslant x \leqslant 1 - \frac{1}{n}\right\}$$

where, $\underbrace{n = 1, 2, 3 \ldots \ldots \ldots}_{\text{index}}$

So, $A_1 = \left[0, 1 - \frac{1}{1}\right] = [0, 1-1] = [0, 0] = [0]$

$A_2 = \left[0, 1 - \frac{1}{2}\right] = [0, 0.5] = [0, 0.5]$

Applying the concept of indexed collection of sets

$$\bigcup_{n=1}^{\infty} A_n = A_1 \cup A_2 \cup A_3 \ldots \ldots \ldots$$

$$= [0, \ldots, 0.5, \ldots 1)$$

→ open interval, the value is always less than 1.

If we apply DeMorgan Laws,

$$\left(\bigcup_{n=1}^{\infty} A_n\right)^c = [0, 1)^c = [1, \infty)$$

Definition: Suppose, $\{A_i, i \in I\}$ is an indexed family of sets, then the union of the family is

$$\bigcup_{i \in I} A_i = \{x \in S : x \in A_i \text{ for at least one } i \in I\}$$

and the intersection is:

$$\bigcap_{i \in I} A_i = \{x \in \underbrace{S}_{\text{sample space}} : x \in A_i \text{ for all } i \in I\}$$

Definition: Collectively Exhaustive

Assume that G is a subset of sample space S. That is, $G \subset S$ and further assume

$$\bigcup_{i \in I} A_i = G$$

Then we can say that the ~~co~~ family of sets $\{A_i ; i \in I\}$ is collectively exhaustive of G.

means that

when we take the unions of the member sets of a family, we get the set G.

A family of sets, denoted as $\{A_i ; i \in I\}$, is disjoint if all pairs are disjoint. Precisely,

$$A_i \cap A_j = \phi, \quad \forall i, j \in I$$

We use collectively exhaustive and disjoint concept to define a partition.

Partition: A family of sets, denoted as $\{A_i ; i \in I\}$ is a partion of the sample space S if it is disjoint and it is collectively exhaustive over the sample space S. That is,

$$\bigcup_{i \in I} A_i = S \quad \text{and} \quad A_i \cap A_j = \phi \ \text{for all } i \neq j$$

✓ The above definition is for a partition over the whole samplespace S. However,

the concept can be extended for any subset G of the sample space S.

For instance, Suppose, $G \subset S$, and $\{A_i ; i \in I\}$ is the family of sets. So, $\{A_i ; i \in I\}$ is the partition of $G \subset S$ if

$$\underbrace{\bigcup_{i \in I} A_i = G}_{\text{collectively exhaustive}} \quad \text{and} \quad \underbrace{A_i \cap A_j = \phi \ \text{for all } i \neq j}_{\text{Disjoint}}$$

⊞ **Fact:** Assume that $\{A_i ; i \in I\}$ is a partition of S, and consider that $G \subset S$. Let's define

$$B_i = A_i \cap G \ , \quad i \in I$$

then $\{B_i ; i \in I\}$ is a $\underbrace{\text{partition of } G}_{\text{to prove}}$.

Proof:
Homework

we have to show that

$\bigcup_{i \in I} B_i = G$ collectively exhaustive

$B_i \cap B_j = \phi$, for $i \neq j$ Disjoint

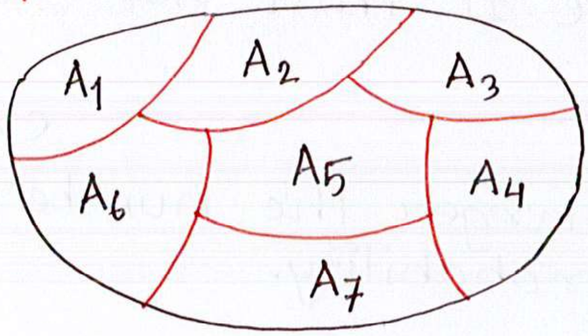
This idea of partion and the related problems are useful for a particular type of decomposition, namely the total probability law.

Very important for the proof of Baye's Theorem and related results.

So, with these basic topics covered from Set Theory we now use those to define the —

Probability Space

Example of a partition



$S = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6 \cup A_7$

For instance, consider $S = \{1, 2, 3, 4, 5, 6\}$, and a collection of sets $A_1 = \{1, 2, 3\}$, $A_2 = \{4, 5\}$, $A_3 = \{6\}$

As observed, A_1, A_2, A_3 are the partition of S

$$A_1 \cup A_2 \cup A_3 = \{1, 2, 3\} \cup \{4, 5\} \cup \{6\} = \{1, 2, 3, 4, 5, 6\}$$

and

$A_1 \cap A_2 = \phi$ | $A_1 \cap A_3 = \phi$

$A_2 \cap A_3 = \phi$

So, our discussion on set-theory terminologies directly relates to probability, and can be summarized as follows:

Set theory	Probability
Set	event
Universal set	Sample space
element	outcome

We can state that

Probability is a number between 0 and 1 that describes a set.

event

Also, larger the number, higher the probability.

Size of the sample space S/Ω

Sample space S of a random experiment, is by definition, is the set of all possible outcomes of the random experiment. For instance,

Finite sample space

- ✓ rolling a fair die has the following sample space $S = \{1, 2, 3, 4, 5, 6\}$
- ✓ Flipping a fair coin has the sample space as follows $S = \{H, T\}$
- ✓ If the coin is flipped thrice, the sample space becomes $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

So, as observed here

- Different random experiments can have different sample space size.
- Size of the sample space S can be finite or infinite.

For instance,

If we flip a coin and the coin is thrown at a different heights, and we are interested to know the speed of the coin when it strikes, the sample space would be $S = \mathbb{R}^+$

Uncountable

whereas,

if we are interested whether its H or T appears in the coin flip, $S = \{H, T\}$ Finite & countable

That is,
the same experiment can have different S depending on our desired outcome or event of interest.

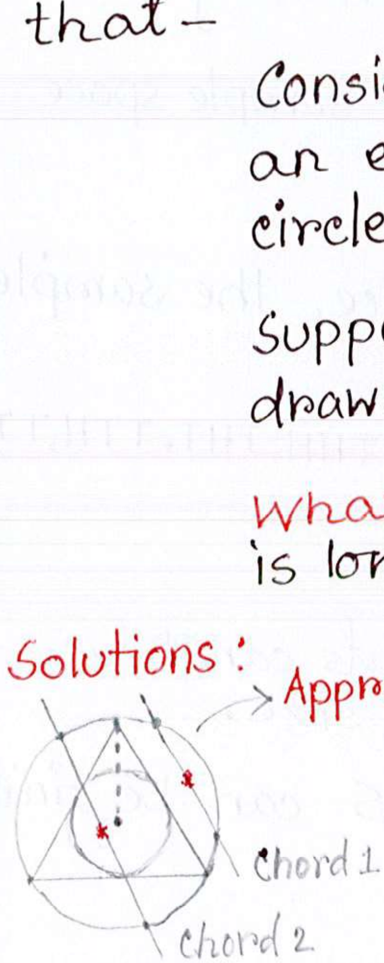
Such scenario can be further observed from the classical **Bertrand Paradox**, which states that –

Consider a circle of radius r. Assume that, an equilateral triangle is inscribed in a circle.

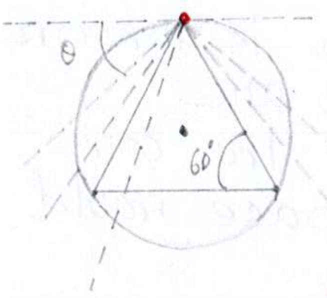
Suppose that a chord of the circle is drawn at random. Then,

What is the probability that the chord is longer than the side of the triangle **?**

Solutions :



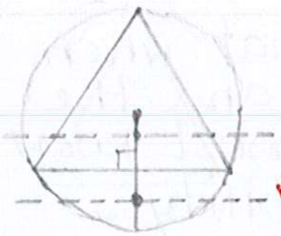
Approach 1 :
- r is the radius
- center of the circle and the centroid of the triangle are same. So, radius of the small circle is $r/2$
- As we see, if the mid-point of the randomly drawn chords (chord 1, chord 2) are inside the smaller (inscribed) circle, the chord would be greater than the side of the triangle. So, the probability would be $\frac{\pi(r/2)^2}{\pi(r)^2} = \frac{1}{4}$



Approach 2 : Select a point on the circle.

equilateral triangle has 60° between two sides. So, when the chord makes an angle $\theta \geq 60^\circ$ & $\theta \leq 120^\circ$, then the chord would be larger than the side of the triangle. So, probability would be

$$\frac{\pi/3}{\pi} = \frac{1}{3}$$



Approach 3: ✓ Consider a radius line, and select a point on the radius.

✓ Draw a chord passing through the point and it is perpendicular to the radius.

✓ Rotate the equilateral triangle and make it parallel to the chord.

✓ The radius and one side of the triangle bisects each other, as they are perpendicular.

Our observation suggests that

When the intersecting point between the radius and the chord is nearer to the center of the circle than the intersecting point between the side of the equilateral triangle and the radius.

So, when the chord intersects the radius at the upper half of the radius. Then the Probality is: $\frac{r/2}{r} = \frac{1}{2}$

For the same question,

we have three logically accurate answers, and these are acceptable answers. But how is it possible?

Answers are different because,

the sample spaces used in these approaches are different.

Approach	Sample space
1	0 to πr^2
2	0 to 2π
3	0 to r

So, the paradox clearly demonstrates that when we talk about the random experiment and the relevant probability of an event, we must know about the exact sample space, probability space.

Also, it is necessary to know if the sample space S is finite or infinite, countable or uncountable — we need set theory again.

Depending on the cardinality of the sample space, they can be classified into the following categories:

Finite sample space

Countably infinite sample space

Uncountable sample space

Thus, it is important to understand the differences between finite and infinite, uncountable and countable

Finite vs. Infinite

A set S is finite with cardinality (number of elements) $n \in \mathbb{N}$ if the set $\{0, 1, 2, 3, \ldots, n-1\}$ has a bijection (one-to-one, onto) to S.

Here, $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ itself is NOT Finite

But, when we take specific number say 100, then we should — one-to-one correspondence.

be able to put the set of our interest, say S, into one-to-one correspondence with set chosen from $\mathbb{N}$, which is

$\{0, 1, 2, \ldots \ldots 99\}$ a finite set.

Simply, A set is finite if it has finite number of elements.

Infinite: A set is infinite if it is not finite.

However, Infinite sets can be classified into two main categories:

- Countable
- Uncountable

Countable: An infinite set is countable if its element can be put into one-to-one correspondence with the elements of the set $\mathbb{N}$.

$\hookrightarrow$ set of counting numbers.

Uncountable: The infite set is uncountable if it can not be put into one-to-one correspondence with the $\mathbb{N}$.

Example: $\mathbb{R} = (-\infty, +\infty)$

$[0, 1]$ and $(0, 1)$

$[a, b]$, $[a, b)$, $(a, b]$, (a, b) for $a < b$

⊞ Show that the odd positive integers is a countable set

To show that odd positive integers is a countable set, we will show that there is a one-to-one correspondence between this set and the set of positive integers/set of counting numbers.

odd positive integers $= \{1, 3, 5, 7, 9, 11, \ldots \ldots\}$

$$\mathbb{Z}^+ = \{1, 2, 3, 4, 5, \ldots \ldots\}$$

If we want to show that a one-to-one fn^c exist between the $\mathbb{Z}^+$ and odd positive integers,

then, all elements of $\mathbb{Z}^+$ must have unique images in the odd positive integers set, for a fn^c.

Let's assume, $f(n) = 2n-1$

1	2	3	4	5	6	7	8	9	10
1	3	5	7	9	11	13	15	17	19

$f(n) = 2n-1$

so, we obtain one-to-one fn^c

For onto, assume that t is any odd integer, and our observation says that t is less by 1 from an even integer. Let's say that even integer is $2K$. So, $t = 2K-1$ for any K integer

$= f(K)$

So, it suggest that all the elements of odd integers must be the image of some elements of $\mathbb{N}$ or $\mathbb{Z}^+$. Hence, $f(n)$ is onto fn^c

Cardinality and countablity

Consider two finite sets:

$A = \{1, 2, 3, 4, 5, 6\}$ | Cardinality $|A| = 6$

$B = \{2, 4, 6, 8, 10, 12\}$ | $|B| = 6$

Simply, by counting the elements in both sets we can say that $|A| = |B| \equiv$ They are equicardinal

Precisely, for finite sets, we can count and compare the total elements of the sets to decide if they have equal cardinality. However, this approach will not work for sets that are infinite.

For instance, if we want to compare between the set of natural numbers $\mathbb{N}$ and set of integers $\mathbb{Z}$ (infinite), the counting approach is not sufficient to compare the two sets $\mathbb{N}$ and $\mathbb{Z}$.

Mathematician Gregor Cantor suggested that using the concepts of Bijective fnc one can compare two sets, even if the sets are Infinite.

This becomes the most efficient way to assess the size of the set, and is very useful for the sample space.

Gregor Cantor's definition on equicardinality

1. Two sets A and B are equicardinal, denoted as $|A| = |B|$, if there's a bijective (also known as one-to-one correspondence) fn^c between A and B (one-to-one, Onto)

2. The cardinality of B is greater than or equal to the cardinality of A ($|B| \geq |A|$) if there exists an injective (one-to-one) fn^c from A to B. That is, if $f: A \to B$ is an injective fn^c

3. The cardinality of set B is strictly greater than the cardinality of set A to B, if there exists an injective fn^c from A to B, but no bijective fn^c from A to B.

We can use the above finition to define the concept of countablity:

Countably infinite: Any set A is countably infinite if A and $\mathbb{N}$ (set of natural numbers) are equicardinal.

A set is countable if it is either finite or countably infinite. So, we need a bijection $f: A \to \mathbb{N}$