Gregor Cantor's definition on equicardinality

1. Two sets A and B are equicardinal, denoted as $|A| = |B|$, if there's a bijective (also known as one-to-one correspondence) fn^c between A and B

   one-to-one
   Onto

2. The cardinality of B is greater than or equal to the cardinality of A $(|B| \geqslant |A|)$ if there exists an injective (one-to-one) fn^c from A to B. That is, if

   $f: A \to B$ is an injective fn^c

3. The cardinality of set B is strictly greater than the cardinality of set A if there exists an injective fn^c from A to B, but no bijective fn^c from A to B.

We can use the above finition to define the concept of countablity:

Countably infinite: Any set A is countably infinite if A and $\mathbb{N}$ (set of natural numbers) are equicardinal.

A set is countable if it is either finite or countably infinite. So,

we need a bijection $f: A \to \mathbb{N}$

□ Show that the set of all integers is countable.

done later

□ Show that the set of positive rational numbers is countable.

The rational numbers set $\mathbb{Q}$ is defined

$$\mathbb{Q} = \{ p/q, \text{ where } p, q \in \mathbb{Z}, \; q \neq 0 \}$$

So, the set of positive rational numbers is defined as: $\mathbb{Q}^+ = \{ p/q, \text{ where } p, q \in \mathbb{N} \}$

p :

q : column

Draw an array as follows:

$$\begin{array}{c|ccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \\
\hline
1 & 1/1 & 2/1 & 3/1 & 4/1 & 5/1 & 6/1 & 7/1 & \cdots \\
2 & 1/2 & 2/2 & 3/2 & 4/2 & 5/2 & 6/2 & 7/2 & \cdots \\
3 & 1/3 & 2/3 & 3/3 & 4/3 & 5/3 & 6/3 & 7/3 & \cdots \\
4 & 1/4 & 2/4 & 3/4 & 4/4 & 5/4 & 6/4 & 7/4 & \cdots \\
5 & 1/5 & 2/5 & 3/5 & 4/5 & 5/5 & 6/5 & 7/5 & \cdots
\end{array}$$

So, the initial terms in the list of positive rational numbers are:

$1, \frac{1}{2}, 2, 3, \frac{1}{3}, \frac{1}{4}, \frac{2}{3}, 4, 5 \ldots$

$r_1, r_2, r_3, r_4, r_5, r_6, r_7 \ldots \ldots r_n$

positive rational numbers are listed as a sequence.

So, the set of positive rational numbers is countable.

Show that the rationals in [0, 1] is countable

We already have shown that positive rationals set is countable. Here, we consider an interval [0,1] and we have to show that rationals in [0,1] is countable.

how to show

we have to show a bijection from [0, 1] to $\mathbb{N}$.

Again, we create an array similar to previous proof as follows:

Let's assume that p/q is a rational number with $p, q \in \mathbb{Z}^+$ and $q \neq 0$

Now list out the number as follows:

$q = 1$, and increase q in step of 1 with value of p between $0 \leqslant p \leqslant q$. 1

List out those p/q as obtained using the previous step.

already in list

We obtain $\left\{0, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \underbrace{\frac{3}{3}}_{=1}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{5}{5} \cdots\right\}$

$\equiv \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \ldots \ldots\right\}$

As we see, we can clearly have a bijection from [0, 1] to $\mathbb{N}$.

So, it is countable

Theorem: Assume that I is a countable index set, and A_i is countable as well for every $i \in I$. Then, union-

$$\bigcup_{i \in I} A_i \text{ is countable}$$

The above theorem can be used to prove the already proven statement

"The set of p sitive rational numbers are countable."

We also can prove if for all the rational numbers. That is:

The set of all Rational numbers is countable.

Proof concept: As we have shown for $[0,1]$, all the rational numbers between $[0,1]$ are countable. That is $\mathbb{Q} \cap [0,1]$ is countable

So,

$$Q_i = \mathbb{Q} \cap [i, i+1]$$

$\mathbb{Q} \cap [1,2]$ is "

$\mathbb{Q} \cap [2,3]$ is "

$\vdots$

We can apply,

$\mathbb{Q} \cap [n, n+1]$ is countable

A countable union of countable sets is countable

$\forall n \in \mathbb{Z}$

$$\bigcup_{i \in \mathbb{Z}} Q_i = \mathbb{Q}$$

where, each Q_i is countable

How to check one-to-one concept for any fn^c ?

For any fn^c, [check that if $f(a) = f(b)$, then it implies that $a = b$

it ensures the unique images for each element.

Consider, $f(n) = 2n-1$ | $f(b) = 2b-1$

so, $f(a) = 2a-1$ | Let, $f(a) = f(b)$

That is, $2a-1 = 2b-1$ | $\Rightarrow a = b$

$\Rightarrow 2a = 2b$

Another example : $f(x) = \frac{x-3}{x+2}$

Assume, a, b are elements in the domain.

$f(a) = \frac{a-3}{a+2}$, $f(b) = \frac{b-3}{b+2}$

Let $f(a) = f(b)$, then $\frac{a-3}{a+2} = \frac{b-3}{b+2}$

$\Rightarrow (a-3)(b+2) = (a+2)(b-3)$

$\Rightarrow \cancel{ab} - 3b + 2a \cancel{-6} = \cancel{ab} + 2b - 3a \cancel{-6}$

$\Rightarrow -3b + 2a = 2b - 3a$

$\Rightarrow 5a = 5b$

so, $a = b$

That is, the fn^c is one-to-one

Another example : $f(x) = 1 - x^2$

Asume that Domain $\mathbb{Z}$ or Co-Domain $\mathbb{R}$

So, $f(a) = f(b)$

$\Rightarrow 1 - a^2 = 1 - b^2$

$\Rightarrow a^2 = b^2$

$\Rightarrow a = \pm b$

that is, if $a = 1$, $b = \pm 1$

so, not one-to-one

Show that the set of integers is countable

$\mathbb{Z}$ = set of integers $= \{0, \pm 1, \pm 2, \pm 3 \dots \dots\}$

As we know, if we can show that $\mathbb{Z}$ can be put into one-to-one correspondence with $\mathbb{N}$, then we can say that $\mathbb{Z}$ is countable.

For instance, one can define the bijection between $\mathbb{Z}$ and $\mathbb{N}$. That is,

$$f : \mathbb{Z} \to \mathbb{N}$$

One way we can put $\mathbb{Z}$ in one-to-one correspondence with $\mathbb{N}$ is the following:

$$\begin{array}{cccccccc} \dots & -3, & -2, & -1, & 0, & 1, & 2, & 3, & \dots \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ & 7 & 5 & 3 & 1 & 2 & 4 & 6 & \end{array}$$

Or, simply as follows:

$$\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \dots \dots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 0 & +1 & -1 & +2 & -2 & +3 & -3 & +4 & \end{array}$$

Probability space:

A probability space $(S, \mathcal{F}(S), P)$ is a triple consists of the three elements as follows:

- The sample space S
- Collection of events $\mathcal{F}$ or denoted as $\mathcal{F}(S)$.
  - * also known as event space
  - * includes the subsets of S
- The probabilities for all the events $A \in \mathcal{F}$

  That is, $P(A), \ \forall A \in \mathcal{F}(S)$, where

  $$P(\cdot) : \mathcal{F}(S) \rightarrow [0, 1]$$

  ↳ acts as a set fn^c.

Let us go over the each element with greater details:

Sample space: S

- sample space of a random experiment is a non-empty set that includes all the possible outcomes of the given random experiment.

- For instance, rolling a fair die has the sample space $\{1, 2, 3, 4, 5, 6\}$

  Flipping a fair coin: sample space $= \{H, T\}$

- sample space is denoted as "S" or "Ω"

- When we perform a random experiment one and only one outcome from S occurs.

Event space

Event space, denoted as $\mathcal{F}(S)$, is a non-empty collection of subsets of S satisfying the three closure properties as stated below:

1. If event $A \in S$, then $\bar{A} \in \mathcal{F}(S)$

   needed, because if we are interested about the occurrence of an event, we would be looking at the event that A does not occur.

2. If for some finite n,

   $$A_i \in \mathcal{F}(S), \quad i = 1, 2, 3, \ldots n$$

   then, $\bigcup_{i=1}^{n} A_i \in \mathcal{F}(S)$

   suggests that

   if $A_1, A_2 \ldots A_n$ are events, then A_1 or A_2 or ... ... or A_n would also be in event space.

3. If $A_i \in \mathcal{F}(S)$, $i = 1, 2, 3, \ldots \ldots n$

   then, $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}(S)$

   This property is for the countable collection of events.

   This is also known as countable unions of collection of countable events.

A set $\mathcal{F}(S)$, which is the event space, satisfying closure properties as stated (property 1, 2, 3) is called a σ-field.

↳ sigma-field

However, if a set $\mathcal{F}(S)$ only satisfies property 1 and property 2, the set is known as Field.

From the definition of sigma-field, it also follows that $\phi \in \mathcal{F}(S)$ and $S \in \mathcal{F}(S)$

Proof: According to the property $A \in \mathcal{F}(S)$ then the complement of A

$$\bar{A} \in \mathcal{F}(S)$$

Using the other property of finite union, as $A, \bar{A} \in \mathcal{F}(S)$, their union would also be in $\mathcal{F}(S)$. So, $A \cup \bar{A} \in \mathcal{F}(S)$

$$\Rightarrow S \in \mathcal{F}(S) \quad \Big| \quad \text{As, } A \cup \bar{A} = S$$

Again, using the property 1, we can say that

$$\bar{S} \in \mathcal{F}(S) \Rightarrow \phi \in \mathcal{F}(S)$$

So, it is shown that $\bar{S} \in \mathcal{F}(S)$, $\phi \in \mathcal{F}(S)$

☐ About intersections: Is $A \cap B \in \mathcal{F}(S)$

Let's say $A \in \mathcal{F}(S)$ $\Rightarrow \bar{A} \in \mathcal{F}(S)$	$B \in \mathcal{F}(S)$ $\Rightarrow \bar{B} \in \mathcal{F}(S)$	Union property $\bar{A} \cup \bar{B} \in \mathcal{F}(S)$

Now,

$$A \cap B = \overline{\overline{A \cap B}} = \overline{\bar{A} \cup \bar{B}}$$

So, $\overline{\bar{A} \cup \bar{B}} \in \mathcal{F}(S)$

Probability measure :

A probability measure $P(\cdot)$ corresponding to the sample space S and event space $\mathcal{F}(S)$ is an assignment of real number (between 0 and 1) to every event $A \in \mathcal{F}(S)$ that satisfies the Axioms of Probability.

Axioms of Probability

From the frequency notion of probability, in any random experiment the probability of an event is the proportion of the time the event occurs in a large number of runs of the experiment. But mathematically this is expressed as per the axioms stated below :

1. $P(A) \geq 0, \quad \forall A \in \mathcal{F}(S)$
2. $P(S) = 1$
3. If any two events A_1 and A_2 are disjoint, then $P(A_1 \cup A_2) = P(A_1) + P_2(A_2)$

By induction process, we can say that if $A_1, A_2 \dots \dots A_n$ are countable collection of events and they are mutually exclusive, then

$$P(A_1 \cup A_2 \cup A_3 \dots \dots \cup A_n) = P(A_1) + P(A_2) + \dots \dots + P(A_n)$$

though $\boxed{3}$ does not imply $\boxed{4}$

We can generalize further, for countable collection

4. If $A_1, A_2 \dots A_n \dots$ are countable collection of disjoint sets, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Probability of particular outcome

Often, we may be interested in the probability of a particular outcome, say $\omega_0 \in S$, occurs in the random experiment.

We do so by calculating the probability of a singleton set $\{\omega_0\}$, which is also known as singleton event. Singleton set means set with only one element. So,

$$\{\omega_0\} \in S \quad \text{and} \quad P(\overbrace{\{\omega_0\}}^{\text{event}}) \text{ is well defined}$$

$$\text{but not, } P(\underbrace{\omega_0}_{\text{outcome}})$$

Examples of (S, F, P)

Consider the sample space of a random experiment that has either "0" or "1" as the probable outcomes. So,

$$\text{Sample space } S = \{0, 1\}.$$

$$\text{Event space } \mathcal{F}(S) = \{\phi, \{0\}, \{1\}, \{0,1\}\}$$

$$\equiv \{\phi, \{0\}, \{1\}, S\}$$

For a complete probability definition, we must assign probability to each event. So,

$$\begin{aligned} P(A) &= \alpha, && \text{when } A = \{0\} \\ &= 1-\alpha, && \text{when } A = \{1\} \\ &= 0, && \text{when } A = \phi \\ &= 1, && \text{when } A = \{0,1\} \end{aligned}$$

where $0 \leqslant \alpha \leqslant 1$

Also, $P(\cdot)$ must satisfy the axioms of probability.

Example 2 : Let's consider a sample space S. Define $\mathcal{F}(S)$, the event space, as $\mathcal{F}(S) = \{\phi, S\}$

By the axioms of probability we can show that only one $P(\cdot)$ will work

$$P(A) = \begin{cases} 1, & \text{when } A = S \\ 0, & \text{when } A = \phi \end{cases}$$

The above $\mathcal{F}(S)$ is called trivial event space.

- Consider an event $B = B_1 \cup B_2 \cup \dots \cup B_m$ and $B_i \cap B_j = \phi$ for $i \neq j$, then

$$P(B) = \sum_{i=1}^{m} P(B_i)$$

Finite union of mutually exclusive sets.

A slight notational modification of the axiom of the probability.

- Furthermore, Assume that an event B includes outcomes as: $B = \{s_1, s_2, \dots \dots s_m\}$

Here, the outcomes are mutually exclusive. So, the probability of the event B is.

$$P(B) = \sum_{i=1}^{m} P(\{s_i\})$$

Prove: each outcome is an event, denoted as $\{s_i\}$ singleton set. So,

$$B = \{s_1\} \cup \{s_2\} \cup \dots \dots \cup \{s_m\}$$

as the outcomes are mutually exclusive.

Using the axioms of probability related to disjont events

$$P(B) = \sum_{i=1}^{m} P(\{s_1\})$$

Recollecting Axioms of Probability

- Roughly speaking, the axiomatic approach to probability was introduced around 1930 by Andre Kolmogorov.

- Assumptions that are made for every probability measure. In fact, these are the minimum set of requirements that probability measure should have

- The axioms :

1. $P(A) \geqslant 0 , \quad \forall A \in \mathcal{F}(S)$

2. $P(S) = 1$ Probability that "something" happens

3. If $A_1, A_2 \in \mathcal{F}(S)$ and $A_1 \cap A_2 = \phi$, then
$$P(A_1 \cup A_2) = P(A_1) + P(A_2).$$

can be extended

If a finite collection of sets $\{A_1, A_2 \dots A_n\}$ are disjoint, then we obtain
$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$
Using Induction process

4. If $A_1, A_2, \dots A_n \dots \in \mathcal{F}(S)$ is a countable collection of disjoint events, then countable union
$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

However, in many books, we may see 1, 2, 4 as the of probability. As, if 4 is true, then 3 is true as well.

□ Example: Let's assume that S is a sample space and $\mathcal{F}(S) \equiv$ event space $= \{\phi, S\}$

According to the axioms of probability, the only probability measure that works for the given event space

$$P(A) = \begin{cases} 1, & \text{when } A = S \\ 0, & \text{when } A = \phi \end{cases}$$

It follows from the axioms of probability that

$$P(\emptyset) = 0$$

Proof: $S = S \cup \phi$, as S, ϕ are disjoint $S \cap \phi = 0$

So, $P(S) = P(S \cup \phi)$

$\Rightarrow P(S) = P(S) + P(\emptyset)$ axiom 3

$\Rightarrow P(S) - P(S) = P(\emptyset)$

$\Rightarrow 1 - 1 = P(\emptyset)$ axiom 1

$\Rightarrow P(\emptyset) = 0$

We extend, it follows from the axioms of probability that $P(\bar{A}) = 1 - P(A)$

Proof: $S = A \cup \bar{A}$, Here, A & $\bar{A}$ are disjoint b definition

$\Rightarrow P(S) = P(A \cup \bar{A})$

$\Rightarrow P(S) = P(A) + P(\bar{A})$ axiom 3

$\Rightarrow 1 = P(A) + P(\bar{A})$

$\Rightarrow P(\bar{A}) = 1 - P(A)$

Let's summarize a few key ideas:

- Random experiment will have random outcomes
- Possible outcomes: S is the set of all possible outcomes
- We have events in random experiment. Events are described as subsets of S
- If $A \subset S$, then we say that event A occurs if the random outcome $\omega \in S$ is in A.
- Events A of interest are collected in the event space $\mathcal{F}(S)$ (6-field)

  satisfies closure properties

- The probability that an event $A \in \mathcal{F}(S)$ occurs is given by $P(A)$,

$$P(\cdot) : \mathcal{F}(S) \rightarrow \mathbb{R}$$

satisfying the axioms of probability.