Let's summarize a few key ideas:

- Random experiment will have random outcomes
- Possible outcomes: S is the set of all possible outcomes
- We have events in random experiment. Events are described as subsets of S
- If $A \subset S$, then we say that event A occurs if the random outcome $\omega \in S$ is in A.
- Events A of interest are collected in the event space $\mathcal{F}(S)$ (6-field)

  satisfies closure properties

- The probability that an event $A \in \mathcal{F}(S)$ occurs is given by $P(A)$,

$$P(\cdot) : \mathcal{F}(S) \rightarrow \mathbb{R}$$

satisfying the axioms of probability.

Numeric Bound on probability

We know from the axioms $P(A) \geq 0$. Let's consider the event is $\bar{E}$. So, $P(\bar{E}) \geq 0$

Now,

$$P(\bar{E}) = 1 - P(E) \quad \text{already shown}$$

$$\Rightarrow P(\bar{E}) = 1 - P(E) \geq 0 \quad \text{Axiom}$$

$$\Rightarrow 1 - P(E) \geq 0 \Rightarrow P(E) \quad 1$$

So, $0 \leq P(E) \leq 1$ Proved

Applying Axioms on random experiment

Consider a random experiment of tossing a coin. So, probable outcome of the experiment is either Head (H) or Tail (T). Thus, the sample space $S = \{H, T\}$

Event space $\mathcal{F}(S) = \{\{H\}, \{T\}, \{H, T\}, \phi\}$

From the application of axioms, we have obtained

$$P(\emptyset) = 0, \quad P(\{H, T\}) = 1$$

As, appearing $\{H\}$ or $\{T\}$ are mutually exclusive

So, $$P(\{H\} \cup \{T\}) = P(H) + P(T) = 1$$

$$\Rightarrow P(\{H\}) + P(\{T\}) = 1$$

Deeper look at uncountable sample space

Let's consider a sample space $S = (\alpha, \beta)$ which is an open interval with $\alpha, \beta \in \mathbb{R}$ and $\alpha < \beta$. So,

$$S = (\alpha, \beta) = \{x \in \mathbb{R} : \alpha < x < \beta\}$$

an uncountable set

Sample space would include any number between α, β, and the sample space would be uncountable.

We are doing a random experiment to pick a number between α, β

Even if we use close interval $[\alpha, \beta] = \{x \in \mathbb{R} : \alpha \leq x \leq \beta\}$ will still remain uncountable.

Another uncountable set is $\mathbb{R}$: set of all real numbers, is also often comes as the sample space. $\mathbb{R} = (-\infty, \infty)$

In many random experiments, $[\alpha, \beta]$ can take a shape like $[0, 1]$, where $\alpha = 1, \beta = 1$

Again, uncountable

Example : Sample space ①

consider the different

$$S = \{\omega_K, \ K = 1, 2, \ldots \ldots n\}$$ Finite

specifically, $S = \{0, 1\}$ or $\{H, T\}$

$S = \{1, 2, 3, 4, 5, 6\}$

rolling a die

Also, sample space S could be countably infinite. ②

For instance, a countable sample space

$$S = \{ \omega_K \, ; \, K = 1, 2, 3, \dots \dots \}$$

$$K \in \mathbb{N}$$

Another one, uncountable ③

$$S = (\alpha, \beta) , \quad \text{where } \alpha, \beta \in \mathbb{R}$$

or, $S = [\alpha, \beta]$; " "

$$\Rightarrow S = \{ x \in \mathbb{R} : \alpha \leqslant x \leqslant \beta \}$$

$\mathbb{R} = (-\alpha, +\alpha)$

Doing any random experiment that generates an outcome between α & β

could be signal strength being affected by random noise

Another sample space

Assume that K-dimensional vectors with co-ordinates from any of the sets as in the examples ①, ②, ③.

Let's form a set with K-dimensional vectors, where the vectors are formed using the elements of sets similar to ①, ②, ③.

Denote those sets using new notations A

so, $S = \underbrace{A \times A \times A \dots \dots \times A}_{\text{Cartesian product of K-fold}}$

$$= \prod_{i=1}^{K} A$$

So, sample space S containing k-dimensional vector requires cartesian product to form the the k-dimensional vector. Such sample space may be useful in areas:

state of a control system, where the control parameters can hold value from set of real numbers. So,

$$A = \mathbb{R}, \text{ and}$$

$$S = \prod_{i=1}^{k} \mathbb{R}$$ if there are k-dimension vectors

Length k binary codeword. For instance, 10100... 0010 type of codeword. Here,

$$A = \{0, 1\}$$

Length k discrete-time signal. If the signal level is denoted as m, then the length k discrete signal be:

$$m_1, m_2, \dots \dots m_k$$

with $A = \mathbb{R}$, as we want any signal value from $\mathbb{R}$.

Example: 5

A sample space of countable sequences drawn form the sets as in example ① to ③

$$S = A \times A \times A \times \dots \dots \times A \times A \dots$$

$$= \prod_{i \in \mathbb{N}} A = \prod_{i=1}^{\infty} A = A^{\mathbb{N}}$$ → set of natural numbers

For instance, If we assume that $A = \{H, T\}$, then $S = \prod_{i=1} \{H,T\} = \{H,T\}^{\mathbb{N}}_{i=1}$

example sequence from S would be of the form

HTHHTH ... ... | We just keep tossing a coin ...

Countable sequence (infinite)

However, if we observe the set of all possible outcomes, the set S will be uncountable, even for a two element set. That is,

Even if a set A is finite, S as evaluated above, will be uncountable.

Why ?

in S

Because, each sequence can be mapped to a point in the interval [0 1]. So, they form a bijection, and hence, are equicardinal.

We already know that [0, 1] interval is Uncountable. So, S is uncountable

Let us consider $S = A^{\mathbb{N}}$, where $A = \{0\ 1\}$

Then, a typical element in S would appear as $\langle a_1, a_2, a_3 \ldots \rangle$, where $a_i \in \{0, 1\}$

with sequence

we can take binary points

with the sequence $\langle a_1, a_2, a_3 \dots \dots \rangle$ we can define binary points:

$$0 \cdot a_1 a_2 a_3 \dots \dots = \sum_{k=1}^{\infty} \frac{a^k}{2^k} \in [0, 1]$$

So, $S = A^{\mathbb{N}}$ can be put into one-to-one correspondence with $[0, 1]$.

It is uncountable

So, $S = A^{\mathbb{N}}$ is uncountable

We can conclude that with the smallest usable set $A = \{0, 1\}$ we have that

$$S = A^{\mathbb{N}} = \prod_{i \in \mathbb{N}} A = A \times A \times A \times \dots \dots$$

is uncountable

Another example: Sample space, 6

Assume that A be any sample space as in previous examples

So, A is assumed to be one of these.

finite $S = \{0, 1\}$ $\{H, T\}$

countable $S = \{\omega_K, K = 1, 2, 3 \dots\}$

Uncountable $\begin{cases} S = (\alpha, \beta) \\ S = [\alpha, \beta] \end{cases}$

Let $S = \prod_{t \in \mathbb{R}} A$ Here, we are considering uncountable cartesian product.

So, for each index we will have different values of A

index is real number

As $t \in \mathbb{R}$, it also represents fractions, so, "what would be the value of A"

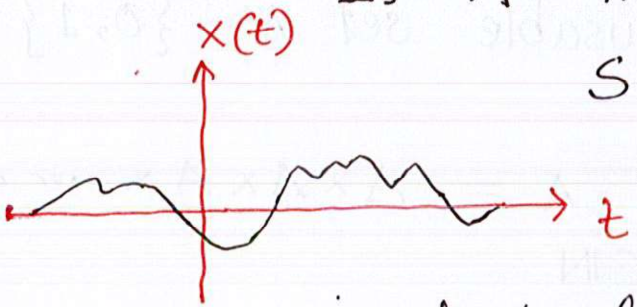
→ The value of A would change as we pick a real number

So,

$$S = \prod_{t \in \mathbb{R}} A = \{\forall \text{ waveforms } x(t),\ t \in (-\infty, \infty) \text{ and } \underbrace{x(t) \in A}_{\text{value of } x(t)},\ \forall t \in (-\infty, \infty)\}$$

→ Set of all waveforms

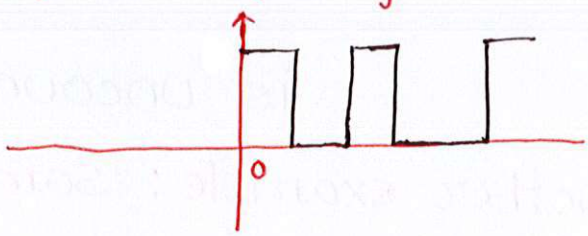
Example: If $A = \mathbb{R}$, it implies that



$S =$ set of all real valued functions of time.

so, it is a real valued fn^c with real variable

if $A = \{0, 1\}$



All the sample spaces considered so far, except the above-one (number 6), are easy to work with.

→ As we have uncountable index, we will have complications.

So, we are done with the examples of sample space S.

Practice Problems

Example

Suppose, we are flipping a fair coin. We want to know that on average, what would be the minimum number of attempts we need to try to 0% sure that we get a head

Answer:

$$\text{Prob. (success after 1 attempt)} = P(H) = \frac{1}{2} = 0{\cdot}5$$

$$\text{Prob. (" " 2 attempts)} = \frac{1}{2} + \left(\frac{1}{2} \times \frac{1}{2}\right)$$

$$= \frac{1}{2} + \frac{1}{4} = 0{\cdot}5 + 0{\cdot}25 = 0{\cdot}75$$

$$\text{Prob (" " 3 attempts)} = \frac{1}{2} + \left(\frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right)$$

$$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 0{\cdot}875$$

Example

What is the probability of getting one tail of 4 independent coin tosses ?

Answer:

$$P(H, , ,T) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$$

$$P(H,H,T,H) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$$

$$P(H,T,H,H) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$$

$$P(T,H,H,H) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$$

So, $P(\text{one tail in 4 coin toss}) = \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}$

$$= \frac{4}{16} = \frac{1}{4}$$

Practice Problems

Example:

Suppose, a computer has 4 processors and a job scheduler selects one processor randomly. What would be the sample space S?

$$S = \{1, 2, 3, 4\}$$

However, If the job scheduler can choose two processors at a time, what would be the sample space?

$$S = \{(1,2), (1,3), (1,4), (2,3) \ldots \ldots (3,4)\}$$

Example: Consider the 'ping' command that we often use see how our internet connections are performing. Precisely, we obtain round-trip time of packets being transmitted from our computers. What would be the sample space for the round-trip times?

$$S = [0, \infty)$$

What is the event that the round-trip time is between 5 ms to 100 ms?

$$E = [5, 100]$$

Example: Not sample space

Let's say we roll a die, and $S = \{1, 2, 3, 4\}$ is not a sample space.

→ Not exhaustive

Examples of event spaces $\mathcal{F}(S)$

Event space:

By definition,

Intuitive understanding

- A collection of events that we are interested in computing the probability of.
- Events are the subsets of sample space S

Mathematically,

Event space $\mathcal{F}(S)$ or $\mathcal{F}$ is a family of subsets of S that satisfy a set of properties, known as closure properties.

Set satisfying these properties are known as σ-field

Closure Properties:

1. If $A \in \mathcal{F}$, then $\bar{A} \in \mathcal{F}$
2. $A_1, A_2 \in \mathcal{F}$, then $A_1 \cup A_2 \in \mathcal{F}$

Follows by induction

If $\underbrace{A_1, A_2, A_3 \cdots A_n}_{\text{Finite collection}} \in \mathcal{F}$, then $\bigcup_{i=1}^{n} A_i \in \mathcal{F}$

3. If $\underbrace{A_1, A_2, A_3 \cdots}_{\text{Countable collection of sets}} \in \mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

Countable Union

If properties 1 and 2 hold for a set collection, it is known as Field of Sets

If properties 1, 2, and 3, all hold for a given set collection, we call it as a sigma field. σ-field

One interesting question would be :

Why don't we construct probability theory using a field of sets instead of a sigma-field (σ-field) of a collection of sets ?

Answer

Probability theory involves results that are expressed in terms of limits of operations on sequences of sets.

we have limit theorems

We compute the probabilities of limits which are expressed as countable sequence of operations on sets.

So, we need countable sequences of set operations to establish these. Limit thrm.

Assured through closure properties.

So, when $\mathcal{P}(\mathbb{R}) = \mathcal{F}$, then there are sets in $\mathcal{F} = \mathcal{P}(\mathbb{R})$ that can not be assigned probability satisfying the Axioms of probability. Thus,

We need to construct a reasonable event space for $S = \mathbb{R}$.

Desired properties of $\mathcal{F}(S)$ when sample space $S = \mathbb{R}$

1. We wish to include all events of the form
$$(a, b) = \{x \in \mathbb{R} : a < x < b\}$$
where $a, b \in \mathbb{R}$ and $a < b$

2. The closure properties of σ-field must be satisfied.

What it means is that,

if all the open intervals are in event space, and the event space satisfies the closure properties of the sigma-field (σ-field), then other events will be in event space

so, we will be able to show that all singleton set of $\mathbb{R}$ are in the event space.

Also, works for close intervals

Definition:

Given S and a family of subsets $G = \{A_i, i \in I\}$ of S, the σ-field generated by G, denoted as $\sigma(G)$, is the smallest σ-field containing all of the subsets in G.

By smallest σ-field, it is meant that

for any σ-field $\mathcal{F}_0$ containing all sets in G,

$$\sigma(G) \subset \mathcal{F}_0$$

So, we can think of all possible sigma-field and take the intersection of those.

So, when $S = \mathbb{R}$

we want the smallest sigma-field that contains all of the open intervals

$$(a,b), \quad a < b, \text{ and } a, b \in \mathbb{R}$$

Precisely, we want $\sigma(G)$, where

$$G = \{(a,b), \forall a, b \in \mathbb{R} \text{ and } a < b\}$$

That is, we need the sigma-field generated by all the open intervals on the $\mathbb{R}$.

will be the event space for $S = \mathbb{R}$

⊞ Comment:

We need the countable sequences of set operations, such as

- Closure under countable unions
- compliment
- Countable intersections through De Morgan's Laws

In general, any countable sequence of set operations involving Union, Intersections, Compliment

These are necessary in order to achieve limiting results as in the limit theorems:

- Weak law of large numbers
- Central limit theorem
- Strong Law of large numbers

So, we need σ-field.

Caution: The Borel Field, as in Papoulis book, is not the same as the sigma field.

All Borel fields are σ-fields, but the reverse is not true.

So, a few examples of event spaces

Example 1: Given a sample space S, an event space $\mathcal{F}(S)$ could be

$$\mathcal{F}(S) = \{\phi, S\}$$

Valid event space

$\phi \in \mathcal{F}(S)$, then $\bar{\phi} \in \mathcal{F}(S)$

$\Rightarrow S \in \mathcal{F}(S)$ & vice-versa

Example 2: Given the sample space S, the set of all subsets, known as the power set, can be d. fined as the event space $\mathcal{F}(S)$.

Inclusion of all the possible subsets in the event space $\mathcal{F}(S)$ makes it also σ-field.

satisfying the closure properties.

However,

- Example 1 is not useful, as it is too small to be useful.

  consider a sample space $[0, 1]$ or $\mathbb{R}$

- Example 2 is not useful as well. Because, for a sample space $[0, 1]$ or $\mathbb{R}$ the event space is too large to be useful.

For $S = \mathbb{R}$, event space $\mathcal{F}(S) = \mathcal{P}(\mathbb{R})$.

where, $\mathcal{P}(\mathbb{R})$ is the power set of $\mathbb{R}$

Precisely, we are not looking anymore at the power set of realnumber set $\mathbb{R}$, we are looking at a smaller version that includes all the events made up of open intervals.

$$G = \{\text{all open intervals}\}$$

So, the σ-field $\sigma(G)$ of $\mathbb{R}$ contains

Smallest σ-field generated by open intervals

- all open intervals → sequence
- countable collections of set operations

$\cup$: unions

$\cap$: Intersections

$\overline{}$: Compliments

on any collection of open intervals.

→ We name it as the Borel-field $\mathcal{B}$

So, given $\mathbb{R}$, the Borel field of $\mathbb{R}$ is defined as the σ-field generated by the open intervals

$$G = \{(a,b) : \forall a,b \in \mathbb{R} \text{ and } a<b\}$$

We denote the Borel-field of $\mathbb{R}$ by $\mathcal{B}(\mathbb{R})$. The members of $\mathcal{B}(\mathbb{R})$ are called Borel sets.

Because, Borel-field contains all open intervals and all countable sequences of operations on these open intervals

So, the Borel-field also contains:

$$(-\infty, b) = \bigcup_{n=1}^{\infty} (b-n\ b) = \lim_{m\to\infty} (-m, b)$$

$$(a, \infty) = \bigcup_{n=1}^{\infty} (a, a+n) = \lim_{m\to\infty} (a, m)$$

$$\{a\} = \bigcap_{n=1}^{\infty} \left(a-\frac{1}{n}, a+\frac{1}{n}\right) = \lim_{n\to\infty} \left(a-\frac{1}{n}, a+\frac{1}{n}\right)$$

Also, applying DeMorgan's Law

$$\bigcap_{n=1}^{\infty} \left(a-\frac{1}{n}, a+\frac{1}{n}\right) = \overline{\overline{\bigcap_{n=1}^{\infty} \left(a-\frac{1}{n}, a+\frac{1}{n}\right)}}$$

$$= \overline{\bigcup_{n=1}^{\infty} \left(a-\frac{1}{n}, a+\frac{1}{n}\right)}$$

$$= \overline{\bigcup_{n=1}^{\infty} \left(a-\frac{1}{n}, a+\frac{1}{n}\right)} \in B(\mathbb{R})$$

Also, $[a,b) = \{a\} \cup (a,b) \in B(\mathbb{R})$

$(a,b] = (a,b] \cup \{b\} \in B(\mathbb{R})$

$[a,b] = \{a\} \cup (a,b) \cup \{b\} \in B(\mathbb{R})$

In addition, all finite and countable sequence of set operations such as $\cup, \cap, -$ of these sets are in $B(\mathbb{R})$.

$B(\mathbb{R})$ includes any subset of $S = \mathbb{R}$ that we would be interested in.