Conditional Probability

In many real-world scenarios, we can not find the precise outcome of experiment. Instead, we can just say that the event B occurs and the precise outcome ω_i is in the set B.

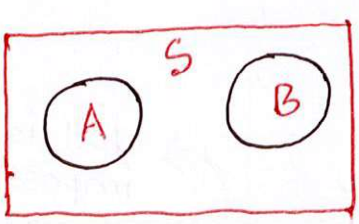
Overall, we just know that event B has occurred.

Assume that two events $A, B \in \mathcal{F}(S)$, and B has occurred.

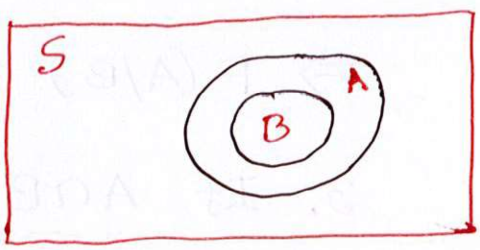
↳ But we don't know the precise outcome.

So, conditional probability describes the probability of occurrence of A, given that the event B has already occurred

That is,
Given $(S, \mathcal{F}, P)$ and $A, B \in \mathcal{F}$ knowing that B has already occurred can tell us about whether the event A occurs or not.



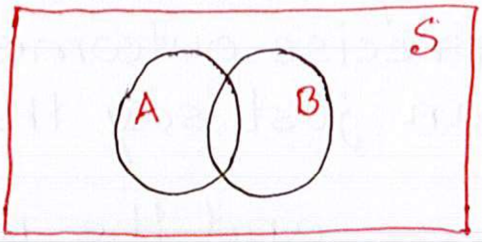
Here, $A \cap B = \phi$. So, if B occurs, then A does not.



Here, $A \cap B = B$, $B \subset A$. So, if B occured, then A must have occurred.

Overall,

Knowledge on whether B has occurred or not may change our belief on the probable occurrence of A.



Definition of $P(A/B)$:

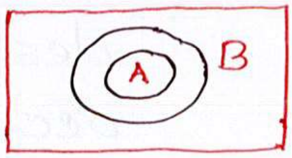
Given the probability space $(S, \mathcal{F}, P)$ and the events $A, B \in \mathcal{F}$, the conditional probability of A conditioned on "B has occurred" is defined as:

$$P(A/B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) \neq 0$$

Probability of A given B

Cases: 1. If $A \subset B$, then $A \cap B = A$, so,

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$



$$\Rightarrow P(A/B) = \frac{P(A)}{P(B)} \geq P(A)$$

2. If $B \subset A$, then $A \cap B = B$, so

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$\Rightarrow P(A/B) = \frac{P(B)}{P(B)}$$

$$\Rightarrow P(A/B) = 1$$

3. If $A \cap B = \phi$, then

Null is an impossible event

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{P(\phi)}{P(B)} = \frac{0}{P(B)} = 0$$

4. From the axioms of probability we know that $0 \leq P(B) \leq 1$

From definition of $P(A/B)$ we impose the constraint that $P(B) > 0$, as the event B must have occurred.

So, we can write for $P(A/B)$ definition that

$$0 < P(B) \leq 1$$

Thus, $P(A/B) = \dfrac{P(A \cap B)}{P(B)} \geq P(A \cap B)$

as, $0 < P(B) \leq 1$

Axiomatic properties of $P(A/B)$

- Axiom 1: $P(A/B) \geq 0 \Rightarrow \dfrac{P(A \cap B)}{P(B)} \geq 0$

  As we know, $P(A \cap B) \geq 0$ and $P(B) > 0$,

  then, $P(A/B)$ should be ≥ 0

- Axiom 2: $P(S/B) = \dfrac{P(S \cap B)}{P(B)} = \dfrac{P(B)}{P(B)} = 1$

  where, $B \subset S$

- Axiom 3: If $A = A_1 \cup A_2 \ldots \ldots$ with $A_i \cap A_j = \phi$ for $i \neq j$, then

$$P(A/B) = P(A_1/B) + P(A_2/B) + \cdots$$

Let's show it for a case where $A = A_1 \cup A_2$ with $A_1 \cap A_2 = \phi$

As, A_1 and A_2 are mutually exclusive, then $A_1 \cap B$ and $A_2 \cap B$ will also be mutually exclusive.

So, $P(A/B) = P\left((A_1 \cup A_2)/B\right)$ distributive law

$$= \frac{P\left((A_1 \cup A_2) \cap B\right)}{P(B)} = \frac{P\left((A_1 \cap B) \cup (A_2 \cap B)\right)}{P(B)}$$

$$= \frac{P(A_1 \cap B) + P(A_2 \cap B)}{P(B)} = \frac{P(A_1 \cap B)}{P(B)} + \frac{P(A_2 \cap B)}{P(B)}$$

$$= P(A_1/B) + P(A_2/B)$$

We can apply mathematical induction to expand the formulation for any n.

So, we can say that

$$(S, \mathcal{F}, P) \xrightarrow[\text{Given}]{\text{B occurred}} (S, \mathcal{F}, P(\cdot/B))$$

Example: Let's calculate the conditional probability $P(A/B)$ for rolling a die experiment. $A = \{2\}$, $B = \{2, 4, 6\}$

As, it is a fair-die, $P(A) = \frac{1}{6}$

$P(B) = \frac{3}{6} = \frac{1}{2}$

So, $P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)}$ as. $A \subset B$

$$= \frac{1}{6} \times \frac{2}{1} = \frac{1}{3}$$

Baye's Theorem and Total probability Law

Assume that $(S, \mathcal{F}, P)$ is a probability space, and A, B are events that are in the event space $\mathcal{F}$. So, $A, B \in \mathcal{F}$.

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$\Rightarrow P(A \cap B) = P(B)\, P(A/B)$$

$$P(B/A) = \frac{P(B \cap A)}{P(A)}$$

$$\Rightarrow P(B \cap A) = P(A)\, P(\ /\ )$$

$$\Rightarrow P(\underbrace{A \cap B}) = P(A)\, P(\ ;\ )$$

Applied commutative law

We can now write

$$P(B)\, P(A/B) = P(A)\, P(B/A)$$

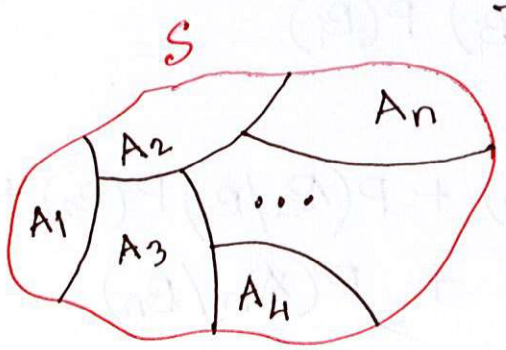
$$\Rightarrow P(A/B) = \frac{P(A)\, P(B/A)}{P(B)}$$

Also known as Bayes Formula

Total Probability Law:

For $(S, \mathcal{F}, P)$, let's assume that $\{A_1, A_2, \dots A_n\}$ be a partition of S, and the event $B \in \mathcal{F}(S)$, then

$$P(B) = P(B/A_1)\, P(A_1) + P(B/A_2)\, P(A_2) + \dots \dots + \dots + P(B/A_n)\, P(A_n)$$



Proof of Total Probability Law

As B is an event and $B \in \mathcal{F}(S)$, we can say that $B \subset S$. So, $B = B \cap S$

Then, $P(B) = P(B \cap S)$

$$\Rightarrow P(B) = P\Big(B \cap \underbrace{\Big(\bigcup_{i=1}^{n} A_i\Big)}\Big)$$

As $\{A_1, A_2 \ldots \ldots A_n\}$ is a partition

$$\Rightarrow P(B) = P\big(B \cap (A_1 \cup A_2 \ldots \cup A_n)\big)$$

$$\Rightarrow P(B) = P\big((B \cap A_1) \cup (B \cap A_2) \cup \ldots \ldots (B \cap A_n)\big)$$

$$= P\Big(\bigcup_{i=1}^{n} (B \cap A_i)\Big)$$

$$= \sum_{i=1}^{n} P(B \cap A_i) \quad \ldots \ldots (1)$$

$A_1, A_2 \ldots$ are mutually exclusive. So, $(B \cap A_1), (B \cap A_2) \ldots$ will be mutually exclusive as well

From the definition of conditional probability

$$P(B / A_i) = \frac{P(A_i \cap B)}{P(A_i)} = \frac{P(B \cap A_i)}{P(A_i)}$$

$$\Rightarrow P(B \cap A_i) = P(B / A_i)\, P(A_i)$$

Using this expression in Eq. 1, we obtain

$$P(B) = \sum_{i=1}^{n} P(B / A_i)\, P(A_i)$$

$$= P(B / A_1) P(A_1) + P(B / A_2) P(A_2) + \ldots \ldots + P(B / A_n)$$

Example: Consider box 1 contains "a" white balls and "b" black balls.

Consider box 2 contains "c" white balls and "d" black balls.

Assume that one ball of unknown color is transferred from box 1 to box 2, and then, one ball is drawn from box 2

Q. What is the probability that the drawn ball from box 2 would be white ball ?

Answer: Let's define $W = \{$transferred ball is white$\}$

From box 1

$$\text{So, } P(W) = \frac{\#\text{ of white balls}}{\#\text{ of total balls}} = \frac{a}{a+b}$$

$B = \{$trasferred ball is black$\} \Rightarrow \frac{b}{a+b} = P(B)$

Now, sample space for box 1 $\equiv \{W, B\}$

Then, $W \cup B = S$, where W, B are disjoint

So, W, B forms a partition.

Consider that the desired event is

$A = \{$White ball is awn from Box 2$\}$

What information is known to us ?

Transferred ball from box 1

↳ unknown color

W, B forms partition

known

$P(W)$

$P(B)$

From Bayes Formula, we know that

$$P(A/B) = \frac{P(B/A)\,P(A)}{P(B)}$$

So, for $A = A_i$,

$$P(A_i/B) = \frac{P(B/A_i)\,P(A_i)}{P(B)} \quad \cdots \cdots (1)$$

Now, using the total probability law we can write for a partition $\{A_1, A_2, \dots A_n\}$ of S

$$P(B) = P(B/A_1)\,P(A_1) + P(B/A_2)\,P(A_2) + \cdots$$

Thus,

$$P(A_i/B) = \frac{P(B/A_i)\,P(A_i)}{P(B/A_1)P(A_1) + P(B/A_2)\,P(A_2) \cdots\cdots}$$

$$= \frac{P(B/A_i)\,P(A_i)}{\sum_{i=1}^{n} P(B/A_i)\,P(A_i)}$$

Bayes Theorem

So, Bayes Theorem is :

Let (S, F, P) is a probability space and $\{A_1, A_2, \dots A_n\}$ is a partition of the sample space S. Consider that $A_1, A_2, \dots A_n \in F$ and event B is in F as well. So,

$$P(A_m/B) = \frac{P(B/A_m)\,P(A_m)}{\sum_{i=1}^{n} P(B/A_i)\,P(A_i)}$$

where, $m = 1, 2, \dots n$

So, $P(A) = P(A \cap S) = P(A \cap (W \cup B))$

$= P((A \cap W) \cup (A \cap B))$ here, $A \cap W$ and $A \cap B$ are disjoint

$= P(A \cap W) + P(A \cap B)$

Using Bayes Formula $P(A \cap W) = P(A/W)P(W)$

$P(A \cap B) = P(A/B)P(B)$

we can write,

$$P(A) = P(A/B)P(B) + P(A/W)P(W)$$

Now, $P(A/W) =$ Probability that white ball is drawn from box 2, given that the transferred ball from box 1 was white

$$= \frac{c+1}{c+d+1}$$

Similarly,
$$P(A/B) = \frac{c}{c+d+1}$$

Then,
$$P(A) = \frac{(c+1)}{(c+d+1)}\frac{a}{(a+b)} + \frac{c}{(c+d+1)}\frac{b}{(a+b)}$$

$$= \frac{a(c+1)+bc}{(c+d+1)(a+b)}$$

$$= \frac{ac+a+bc}{(c+d+1)(a+b)}$$

Answer.

Problem : Suppose, there are three types of players in a tournament. A, B, C

A is 50 % | C is 25%

B is 25% |

Your winning chance against A, B, C types are 0·3, 0·4, 0·5 respectively.

Assume that, You play a match, so,

a) What is the probability that you win the match ?

b) What is the probability that you played against A-type player ? Given, You won

Answer: $P(A) = 0{\cdot}5,\ P(B) = 0{\cdot}25,\ P(C) = 0{\cdot}25$

Let's define $W \equiv$ You win in the match.

So,

$$P(W/A) = 0{\cdot}3$$
$$P(W/B) = 0{\cdot}4$$
$$P(W/C) = 0{\cdot}5$$

A, B, C are the players type and they are disjoint. Also, their union will give all the player types

Partition Concept

Then,

$$P(W) = P(W/A)\,P(A) + P(W/B)\,P(B) + P(W/C)\,P(C)$$

$$= (0{\cdot}3 \times 0{\cdot}5) + (0{\cdot}4 \times 0{\cdot}25) + (0{\cdot}5 \times 0{\cdot}25)$$

$$= 0{\cdot}375$$

Answer

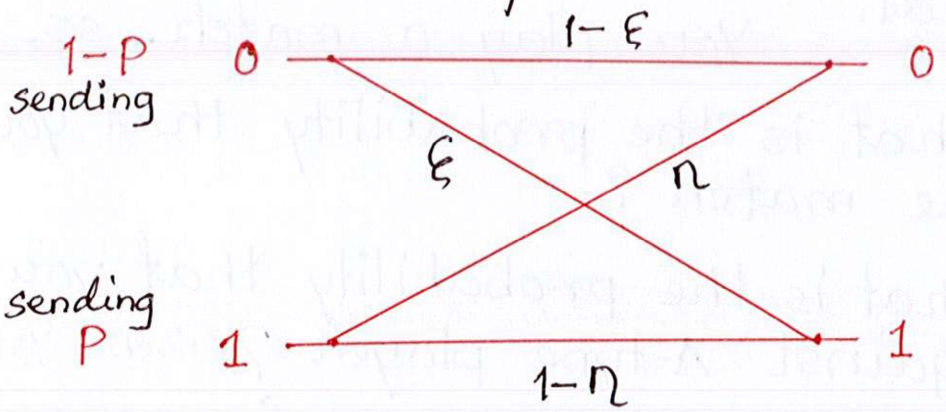
Total probability Law

We have to find out $\quad P(A/W) = ?$

Using Bayes Formula

$$P(A/W) = \frac{P(W/A)\,P(A)}{P(W)} = \frac{0{\cdot}3 \times 0{\cdot}5}{0{\cdot}375}$$

Q. Consider a binary communication channel as:



As we see, Probability of sending 1 is p

Probability of sending 0 is $q = 1-p$

Given that 1 is sent,

Probability of receiving 1 is $1-\eta$

Probability of receiving 0 is η

Given that 0 is sent

Probability of receiving 0 is $1-\xi$

Probability of receiving 1 is ξ

So, what is the probability that 1 has been correctly received?

Answer: We can not consider that $1-\eta$ as the sole probability.

Because, 1 can be received because of a 0 transmission as well.

As transmitting either zero or one is mutually exclusive, we can use the concept of partition

So, $S_0 = 0$ is sent | $R_0 = 0$ is received
$S_1 = 1$ is sent | $R_1 = 1$ is received

Using total probability law

$$P(R_1) = P(R_1/S_1)\,P(S_1) + P(R_1/S_0)\,P(S_0)$$
$$= (1-\eta)P + \varepsilon(1-P)$$

Let's extend it further

suppose we have received 1.

What is the probability that 1 was originally sent?

That is, we are interested in,

$$P(S_1/R_1) = \frac{P(R_1/S_1)\,P(S_1)}{P(R_1)}$$
$$= \frac{(1-\eta)P}{(1-\eta)P + \varepsilon(1-P)}$$

Q. Simplify the countable collection of unions

$$\bigcup_{n=1}^{\infty} [5,\ 8-(2n)^{-1}] = \bigcup_{n=1}^{\infty} [5,\ 8-\tfrac{1}{2n}]$$

$$= [5,\ 8-\tfrac{1}{2}] \cup [5,\ 8-\tfrac{1}{4}] \cup [5, 8-\tfrac{1}{6}] \cdots\cdots$$

A $\subset$ B $\subset$ C $\subset$ ...

$= \quad [5, 8] \qquad$ As, $n \to \infty$, $\frac{1}{2 \times n} \cong 0$

so, $8-0 = 8$

Q. Suppose, You import devices from three different sources:

	Source	defective
Proportion of devices you import from different sources	A	0.005
	B	0.001
	C	0.01

A : 0.5 | C : 0.4
B : 0.1

a) Probability that a randomly selected device is defective and that it is from source A is: $P(D/A) = ?$

$$P(D) = P(D/A)P(A) + P(D/B)P(B) + P(D/C)P(C)$$

$$= (0.005)\,0.5 + (0.001)\,0.1 + (0.01)\,0.4$$

$$= 0.0066$$

$$P(A/D) = \frac{P(D/A)\,P(A)}{P(D)} = \frac{0.00 \quad \times 0.5}{0.0066}$$

$=$

b) $$P(B/D) = \frac{P(D/B)\ P(B)}{P(D)}$$

$$= \frac{0{\cdot}001 \times 0{\cdot}}{0{\cdot}0066} = 0{\cdot}0152$$

c) $$P(C/D) = \frac{P(D/C)\ P(C)}{P(D)} = \frac{0{\cdot}01 \times 0{\cdot}4}{0{\cdot}0066} = 0{\cdot}6$$