Statistical Independence

Definition:

Given $(S, \mathcal{F}, P)$, assume that $A, B \in \mathcal{F}$ then the events A and B are statistically independent if and only if $P(A \cap B) = P(A)\,P(B)$

In a simpler notion, we can say that Two events are independent if $P(A \cap B) = P(A)P(B)$

$$\Rightarrow P(AB) = P(A)P(B)$$

Properties:

Given that, A and B are statistically independent, then

$\bar{A}$ and B are independent

A and $\bar{B}$ are independent

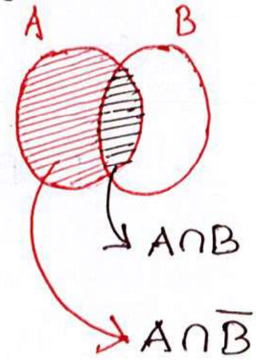
$\bar{A}$ and $\bar{B}$ are independent

Proof: As A, B are statistically independent

$$P(A \cap B) = P(AB) = P(A)\,P(B)$$

we have to show that $P(A \cap \bar{B}) = P(A)\,P(\bar{B})$

Let's write A as the disjunction of two mutually exclusive sets.



$$A = (A \cap B) \cup (A \cap \bar{B})$$

We used $P(A) = 1 - P(\bar{A})$ form of formula

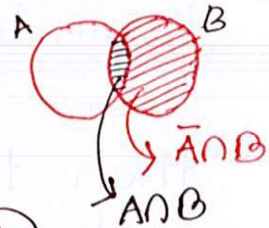
$$\Rightarrow P(A) = P(A \cap B) + P(A \cap \bar{B})$$

$$\Rightarrow P(A) = P(A)P(B) + P(A \cap \bar{B})$$

$$\Rightarrow P(A) - P(A)P(B) = P(A \cap \bar{B})$$

$$\Rightarrow P(A)\underbrace{\left[1 - P(B)\right]}_{P(\bar{B})} = P(A \cap \bar{B}) = P(A)\,P(\bar{B})$$

Again, we can write B as a disjunction of two sets: $B = (A \cap B) \cup (\bar{A} \cap B)$



$$\Rightarrow P(B) = P(A\cap B) + P(\bar{A}\cap B)$$
$$\Rightarrow P(B) = P(A)\,P(B) + P(\bar{A}\cap B)$$
$$\Rightarrow P(B) - P(A)\,P(B) = P(\bar{A}\cap B)$$
$$\Rightarrow P(B)\left[1 - P(A)\right] = P(\bar{A}\cap B)$$
$$\Rightarrow \underbrace{P(B)\,P(\bar{A})} = P(\bar{A}\cap B)$$

shows that they are statistically independent

□ Consider that we toss a coin twice

Assume, $P(H) = a$, $P(T) = b$ and
$a + b = P(H) + P(T) = 1$

Let's consider

$$E_1 = \{H \text{ at first toss}\}$$
$$E_2 = \{H \text{ at second toss}\}$$

show that the events are statistically independent.

Answer: sample space for two consecutive tosses $S = \{HH, HT, TH, TT\}$

So, $E_1 = \{HH, HT\}$, $E_2 = \{HH, TH\}$

$P(E_1) = P(HH) + P(HT)$	$P(E_2) = P(HH) + P(TH)$
$= a^2 + ab$	$= a^2 + ba$
$= a(a+b)$	$= a(a+b)$
$= a$	$= a$

Now, $P(E_1 \cap E_2) = P(E_1)\,P(E_2) = a$

Also, $E \cap E_2 = \{HH\}$ and $P(HH) = a^2$

Independence of three events

The three events A_1, A_2, A_3 are statistically independent

if they are independent in pairs

$$P(A_1 \cap A_2) = P(A_1)\, P(A_2)$$
$$P(A_2 \cap A_3) = P(A_2)\, P(A_3)$$
$$P(A_3 \cap A_1) = P(A_3) \cap P(A_1)$$

and

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)\, P(A_2)\, P(A_3)$$

Also, we can write

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)\, P(A_2)\, P(A_3)$$
$$= P(A_1)\, P(A_2 A_3)$$

So, any single event is independent of the intersection of the other two.

Q. What about $P(A_1 \cap A_2 \cap \bar{A}_3)$?

$$A_1 A_2 = A_1 A_2 A_3 \cup A_1 A_2 \bar{A}_3$$
$$\Rightarrow P(A_1 A_2) = P(A_1 A_2 A_3) + P(A_1 A_2 \bar{A}_3)$$
$$= P(A_1 A_2)\, P(A_3) + P(A_1 A_2 \bar{A}_3)$$

$$\Rightarrow P(A_1 A_2 \bar{A}_3) = P(A_1 A_2) - P(A_1 A_2) P(A_3)$$

So, replacing A by $\bar{A}$ results that the events are independent

$$= P(A_1 A_2)\left[1 - P(A_3)\right]$$
$$= P(A_1 A_2)\, P(\bar{A}_3)$$
$$= P(A_1)\, P(A_2)\, P(\bar{A}_3)$$

Answer

We can generalize our observation on statistically independent events. For instance, let's consider that we have "n" events, and we have been able to define independence of K events for every possible $K<n$.

Then, we can say that events $A_1, A_2, \dots \dots A_n$ are independent if any $K<n$ of them are independent. So,

$$P(A_1, A_2, \dots A_n) = P(A_1)\, P(A_2) \dots P(A_n)$$

Combined Experiments

Consider two experiments

- Rolling a fair die

  $S_1 =$ Sample space $= \{1, 2, 3, 4, 5, 6\}$

  $P(\text{any face of the die}) = \frac{1}{6}$

- Flipping a coin

  $S_2 = \{H, T\}$, $P(H) = P(T) = \frac{1}{2}$

For instance, we perform both experiments and we want the probability that we get two on the die and "H" on the coin.

How do we achieve it ?

We can combine the experiment and form a new sample space.

Suppose, we have two random experiments $(S_1, \mathcal{F}_1, P_1)$ and $(S_2, \mathcal{F}_2, P_2)$

To combine them, we take cartesian product between sample space and form a new sample space

$$S = S_1 \times S_2$$

Here, elements of S are ordered pairs $(a, b) \in S$ with

$$a \in S_1 \text{ and } b \in S_2$$

Based on the two experiments of rolling a die and flipping a coin, we can form a combined experiment.

So, $S = S_1 \times S_2 = \{(H,1),(H,2),(H,3),(H,4),(H,5)$
$(H,6),(T,1),(T,2),(T,3),(T,4),(T,5),(T,6)\}$

$\boxed{+}$ An event in the new combined experiment will be a subset of the sample space

$$S = S_1 \times S_2$$

Now, if $A \subset S_1$ and $B \subset S_2$, then

$C = A \times B$ is a subset ($\subset$) of S and will be an event in the new event space $\mathcal{F}$.

For instance, $A = \{H\}$, $B = \{3, 6\}$, we have

$A \times B = \{(H,3),(H,6)\}$ which certainly is in the new event space $\mathcal{F}$, as $\{(H,3),(H,6)\} \subset S$.

However, not all events can be written as the cartesian products.

How will we then form the event space in the combined experiment?

Answer would be, to form the σ-field generated by all cartesian products.

$$\sigma(\{A \times B : \forall A \in \mathcal{F}_1 \text{ and } \forall B \in \mathcal{F}_2\})$$

This is going to be the event space for the combined experiment

Interestingly, the closure properties of σ-field takes care of the events that cannot be written as the cartesian product of the events from F_1 and F_2

For instance, $S_1 = \{1, 2, 3, 4, 5, 6\}$

$S_2 = \{H, T\}$

Define an event $C = \{H\} \times \{2, 5\} = \{(H,2), (H,5)\}$

another event $D = \{H, T\} \times \{2\} = \{(H,2), (T,2)\}$

Let's define a new event E of the form

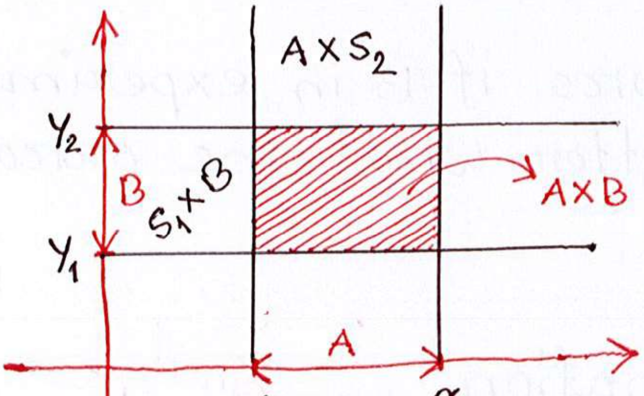
$$E = C \cup D = \{(H,2), (H,5)\} \cup \{(H,2), (T,2)\}$$

$$= \{(H,2), (H,5), (T,2)\}$$

Cartesian Product of two arbitrary sets

Let's assume two sets $A = \{x_1 \leq x \leq x_2\}$

$B = \{Y_1 \leq Y \leq Y_2\}$



As we see, $A \times B$ is the rectangle, and it forms from the two trip.

$A \times S_2$: Vertical strip

$S_1 \times B$: Horizontal strip

Generally, cartesian product of any two arbitrary sets as a generalized rectangle.

Let's consider S_1 and S_2 are two experiments and we form a new experiment $S = S_1 \times S_2$.

The events from experiment S are all cartesian products of the form: $A \times B$

where, A is an event of S_1
B is an event of S_2

So, How do we assign probability to a combined set?

Probabilities of the events $A \times S_2$ and $S_1 \times B$ are

also, known as consistency condition

$$\left[\begin{aligned} P(A \times S_2) &= P_1(A) \\ P(S_1 \times B) &= P_2(B) \end{aligned}\right]$$

Assuming that S is a combined experiment

From the rectangular representation we can say that.

Event $A \times S_2$ occurs if A in experiment S_1 occurs no matter what the outcome in S_2 is.

Event $S_1 \times B$ occurs if B in experiment S_2 occurs no matter what the outcome in S_1 is.

With all these information,

combined event

How do we define $P(C)$ for other events $C \in \mathcal{F}$?

event space for combined event

A few known information for $P(C)$:

$P(C)$ must satisfy the axioms of probability

$P(C)$ must satisfy the consistency conditions

Generally,

The probabilities of events of the form $A \times B$ and of their unions and intersections can not be expressed in terms of P_1 and P_2.

→ We need additional information about S_1 and S_2

Independent Experiments

Often in many experiments, the events $A \times S_2$ and $S_1 \times B$ of the combined experiment S are independent for any A and B. So,

considering the independency,

$$
\begin{aligned}
P(A \times B) &= P(A \times S_2) \cap P(S_1 \times B) \\
&= P(A \times S_2)\, P(S_1 \times B) \\
&= P_1(A)\, P_2(B)
\end{aligned}
$$

To summarize,

$$A \times S_2 \perp\!\!\!\perp S_1 \times B \quad \forall A \in \mathcal{F}_1,\ \forall B \in \mathcal{F}_2 \quad \text{Then,}$$

$(S_1, \mathcal{F}_1, P_1)$ and $(S_2, \mathcal{F}_2, P_2)$ are independent

So, as we obtain, for two independent experiments $(S_1, \mathcal{F}_1, P_1)$ and $(S_2, \mathcal{F}_2, P)$ we have a combined experiment $(S, \mathcal{F}, P)$ with

$$S = S_1 \times S_2$$

$$\mathcal{F} = \sigma(\{A \times B \mid \forall A \in \mathcal{F}_1 \text{ and } \forall B \in \mathcal{F}_2\})$$

$$P(A \times B) = P_1(A) \cdot P_2(B)$$

where, axioms of probability fill in the probabilities of events that can not be written as cartesian products

But those events can be written as a union of disjoint cartesian products.

We can generalize it for n independent experiments:

consider $(S_1, \mathcal{F}_1, P_1), (S_2, \mathcal{F}_2, P_2) \dots (S_n, \mathcal{F}_n, P_n)$ be n random experiments.

Let's form the combined experiment as

$$S = S_1 \times S_2 \times S_3 \times \dots \dots \times S_n$$

$$\mathcal{F} = \sigma(\{A_1 \times A_2 \times \dots \dots A_n \ \forall A_1 \in \mathcal{F}_1, \forall A_2 \in \mathcal{F}_2 \dots \forall A_n \in \mathcal{F}_n\})$$

σ-field

Then,

$$P(A_1 \times A_2 \times \dots \dots \times A_n) = P(A_1) \cdot P(A_2) \cdot P(A_3) \dots \dots P(A_n)$$

and axioms of probability take care of the rest.