Special Case: Bernoulli Trials

Consider n objects, the number of permutations possible are (in orders)

$$n(n-1)(n-2)\ldots\ldots 3.2.1 = n!$$

Suppose, we are interested about taking k object out at a time, and we consider order in each such group.

# of ways	n	$n-1$	$n-2$	$n-(k-1)$	Total number of distinet arrangements of n objects taken k at a time is:
position	• 1	• 2	• 3	• k	

$$n(n-1)(n-2)\ldots(n-k+1) = \frac{n!}{(n-k)!}$$

For instance, if we have 3 objects: a, b, c by taking two objects, the number of permutations

ab, bc, ca, ba, ac, cb $\equiv 6$

Using the formula: $n = 3$, $k = 2$ | $\frac{3!}{(3-2)!} = \frac{6}{1!} = \frac{6}{1}$

$= 6$

Now, if k objects are taken out of n objects without paying attention, then a situation suggests that

ab $\equiv$ ba
bc $\equiv$ cb
ca $\equiv$ ac

In general, without the attention to order taking out k objects from n objects become a combination problem, and is given by

$$\frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = \frac{n!}{(n-k)!\,k!} = \binom{n}{k}$$

Now, Let's consider

a simple experiment $(S_0, \mathcal{F}_0, P_0)$

Assume that, we perform this experiment independently for n-times.

1^{st}-time $\quad 2^{nd} \quad \ldots \quad n^{th}$-time

$$(S_1, \mathcal{F}_1, P_1),\ (S_2, \mathcal{F}_2, P_2),\ \cdots\cdots\cdots (S_n, \mathcal{F}_n, P_n)$$

where,

$$S_1 = S_2 = S_3 \cdots = S_n = S_0(\cdot)$$

$$\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3 \cdots = \mathcal{F}_n = \mathcal{F}_0(\cdot)$$

$$P_1(\cdot) = P_2(\cdot) = P_3(\cdot) \cdots \cdots = P_n(\cdot) = P_0(\cdot)$$

When can this happen ?

For example, tossing a fair-coin for n-times, independently.

So, $(S_1, \mathcal{F}_1, P_1)$ will explain the probability space of the first toss of the coin, and it follows for the rest.

We want to combine the trials and form the combined experiment of these n trials.

We form $(S, \mathcal{F}, P)$ with

$$S = S_1 \times S_2 \times \dots \dots \times S_n$$

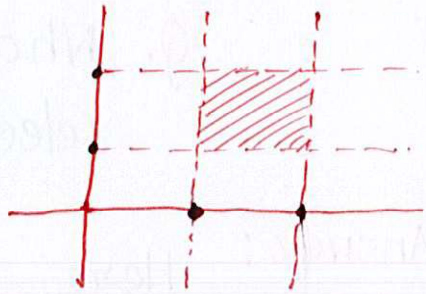
$$\mathcal{F} = \sigma(\{\text{all of the } \underbrace{\text{cylinder sets}}_{\text{forms from Cartesian product}}\})$$

Assume that,

we have an event $A \in \mathcal{F}$

we know the probability

$$P_0(A) = p$$



When the experiment is repeated n-times, we want to know the probability of event $B_K \in \mathcal{F}$ defined as

B_K = A occurs exactly K-times in independent trials/repetitions of $(S_0, \mathcal{F}_0, P_0)$

So,

$$P(B_K) = P_n(K)$$

Theorem: $P_n(K) = \binom{n}{K} p^K (1-p)^{n-K}, \quad K = 0, 1, 2 \dots n$

where, $p = P_0(A)$

which is, Binomial pmf

Example: A box contains two coins, a fair coin and a strange coin.

For the fair coin $P(H) = P(T) = \frac{1}{2}$

For the strange coin $P(H) = \frac{3}{4}$, $P(T) = \frac{1}{4}$

Let's select a coin at random and flip it 4 times. It comes up three heads

Q. What is the probability that I selected the "strange coin"

Answer: Here, $S_1 = \{F, S\}$ F: Fair, S: Strange

$S_2 = \{HHHH, HHHT, HHTH, \ldots \ldots HTTT, TTTT\}$

So, combined experiment will have a sample space $\boxed{S_1 \times S_2}$ Let's define, I select a coin and then I toss that coin 4 times

event F = fair coin is selected

event S = strange coin is selected

A_3 = 3 "Heads" in 4 tosses

We have to calculate $P(S|A_3) = ?$

We know that

$$P(S/A_3) = \frac{P(S \cap A_3)}{P(A_3)}$$

$$= \frac{P(A_3/S)\,P(S)}{P(A_3)}$$

Here, $P(S) = P(F) = \frac{1}{2}$. so,

$$P(A_3|S) = \binom{4}{3} p^3 (1-p)^1,$$ where p is the prob. of single success.

$$= \binom{4}{3}\left(\frac{3}{4}\right)^3\left(1-\frac{3}{4}\right)$$ $\hookrightarrow P(H) = \frac{3}{4}$

$$= \binom{4}{3}\left(\frac{3}{4}\right)^3 \cdot \frac{1}{4} = \frac{27}{64}$$

Again,

$$P(A_3) = P(A_3 \cap \overset{\text{Whole space of fair and strange}}{S^*}) = P(A_3(\text{Strange} \cup \text{fair}))$$

$$= P(A_3 \cap (S \cup F))$$

$$= P((A_3 \cap S) \cup (A_3 \cap F))$$

$$= P(A_3 \cap S) + P(A_3 \cap S)$$

$$\Rightarrow P(A_3) = P(A_3/S)P(S) + P(A_3/F)P(F)$$

$$\Rightarrow P(A_3) = \frac{27}{64} \times \frac{1}{2} + \underbrace{\binom{4}{3}\left(\frac{1}{2}\right)^3 \frac{1}{2}}_{\text{Again, Bernouli trial}}$$

$$= \frac{27}{128} + \frac{1}{8} = \frac{43}{128} = P(A_3)$$

So, $$P(S/A_3) = \frac{P(A_3/S)\,P(S)}{P(A_3)} = \frac{\left(\frac{27}{64}\right)\left(\frac{1}{2}\right)}{\left(\frac{43}{128}\right)}$$

Similarly,

$$= \frac{27}{43} \cong 0.6279$$

$$P(F/A_3) = \frac{P(A_3/F)}{P(A_3)} = \frac{16}{43} \cong 0.3721$$

Classical Probability

In classical probability we have probability space defined as $(S, \mathcal{F}, P)$, where

$|\ |$ cardinality $\equiv$ No. of elements

S : Sample space, and it is finite $|S| = n$

$\mathcal{F}$: Event space $|\mathcal{F}| = 2^n$

All the outcomes are equally Likely. That is, we have pmf : $P(\omega) = \frac{1}{n} \ \forall \omega \in S$

So,

$$P(A) = \sum_{\omega \in A} P(\omega) = \frac{|A|}{n} = \frac{|A|}{|S|}$$

Then, according to classical probability, we have

$$P(\emptyset) = \frac{|\emptyset|}{|S|} = \frac{0}{n} = 0$$

$$P(S) = \frac{|S|}{|S|} = 1$$

When $A, B \in \mathcal{F}$ are disjoint $(A \cap B = 0)$, then,

$$P(A \cup B) = \frac{|A \cup B|}{|S|} = \frac{|A| + |B| - |A \cap B|}{|S|}$$

$$= \frac{|A|}{|S|} + \frac{|B|}{|S|} - \frac{|A \cap B|}{|S|} = 0$$

$$= \frac{|A|}{|S|} + \frac{|B|}{|S|} = P(A) + P(B)$$

If $A, B \in \mathcal{F}$ are not disjoint

$$P(A \cup B) = \frac{|A \cup B|}{|S|} = \frac{|A| + |B| - |A \cap B|}{|S|}$$

$$= \frac{|A|}{|S|} + \frac{|B|}{|S|} + - \frac{A \cap B}{|S|} = P(A) + P(B) - P(A \cap B)$$

Difficulties in classical probability

Assume, A: event in $\mathcal{F}$ | S: Finite

In classical probability: | $|S| = n$ ↳ Cardinality of "S"

- We always assume that the sample space is finite
- Event space is naturally the power set of sample space, as we have finite sample space.

$$\mathcal{F} = \mathcal{P}(S), \quad |\mathcal{F}| = 2^n = 2^{|S|}$$

- All the outcomes are equally likely, which indicates that

$$P(\omega) = \frac{1}{n}$$

[n objects/outcomes & they are equally likely]

So, $p(\omega) = \frac{1}{n} \; \forall \omega \in S$

↳ Small p is for probability of occurrence

Using the probability of individual outcomes we can calculate the probability of any event as:

$$P(A) = \text{Probability of event } A$$

$$= \sum_{\omega \in A} P(\omega) = \frac{|A|}{n} = \frac{|A|}{|S|}$$

Difficulties arise because of the calculation of $|A|$ becomes harder when the counting itself for classical probability is difficult.

Combinatorial counting techniques may be necessary. complex

We already have seen that

$$P(A \cup B) = \frac{|A \cup B|}{|S|} = \frac{|A| + |B| - |A \cap B|}{|S|}$$

$$= \frac{|A|}{|S|} + \frac{|B|}{|S|} - \frac{|A \cap B|}{|S|}$$

$$= P(A) + P(B) - P(A \cap B)$$

$\equiv$ expression we obtained for axiomatic probability

We can extend the outcome to general finite unions:

If events $A_1, A_2, \dots A_n \in \mathcal{F}$ and they are disjoint ($A_i \cap A_j = \phi$, $i \neq j$), then

$$A_i \cap A_j = \phi, \quad \forall i \neq j$$

Then,

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum |A_i|$$

# of elements in unions of A_i

sum of # of elements in each A_i

So, in classical probability definition, if $A_1, A_2, \dots A_n$ are disjoint, then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \frac{\left|\bigcup_{i=1}^{n} A_i\right|}{|S|} = \sum_{i=1}^{n} \frac{|A_i|}{|S|} = \sum_{i=1}^{n} P(A_i)$$

Also,

For $|\bar{A}| = |S-A|$, we can calculate

$$P(\bar{A}) = \frac{|\bar{A}|}{|S|} = \frac{|S-A|}{|S|} = \frac{|S|}{|S|} - \frac{|A|}{|S|}$$

$$= 1 - P(A) \equiv \text{Axiomatic Definition}$$

Overall, the above discussion demonstrates that classical probability measure behaves in a similar way an axiomatic probability measure behaves.

In general,

notion of probability has alternative perspective. For instance,

1. Classical 2. Empirical
3. Subjective 4. Axiomatic

Classical :

- Alternatively, often termed as "a priori" or "theoretical"

rolling a fair die

- Consider a random process where there are n equally likely outcomes and the event A consistst of m of those n outcomes

Then, $P(A) = \frac{m}{n}$, where $m = |A|$, $n = |S|$

For instance, outcomes $\{1\}, \{2\}, \{3\}, \{4\}$ $\{5\}, \{6\}$ are equally likely.

So, $P(1) \equiv P(\{1\}) = \frac{1}{6} = P(2) = P(3) = P(4) = P(5) = P(6)$ $P(\{*\})$

Let's define an event A as:

$$A = \{\text{outcome is even number}\} = \{2, 4, 6\}$$

So, $P(A) = \frac{|A|}{|S|} = \frac{3}{6} = \frac{1}{2}$

- Classical notion is simple for many situations, where outcomes are equally likely
- But the usefulness of the classical notion is limited.

  For instance, if we design an experiment of rolling a die with weighted die, then the outcomes are not equally likely.
- Classical notion also doesn't comply with situations where the outcomes are not finite. One such example would be, for instance, studying stock-prices, where possible price, in theory, may be of any quantity.

Empirical : Also known as "Frequentists"

- The empirical perspective defines probability via thought experiment.

- For instance, consider a weighted die, where the appearance of the faces are not equally likely.

  So, we can obtain the rough idea on the probability of each outcomes by rolling it for a large number of times.

- Now, we can use the proportion of times the die gives a specific outcome, to estimate its probability of occurrence

  specific outcome

- Interestingly, accuracy of probability measure depends on the the number of trials done. That is, if the die is rolled 100 times and 1000 times on separate experiment, probability measure obtained through 1000 rolls will be more accurate.

  So, the true probability is the limit of the approximation, where increasing the number of experiments.

It also works for equally likely outcomes. | Large "n" is a problem | Say, number of tosses

Subjective notion of probability :

- Depends on how as an observer we measure the probability that an event will occur.

  For instance, our personal belief or measure if the price of stock will increase tomorrow.

  Possible to be in agreement with classical or empirical views

- Limitation : As it is subjective, probability might differ between persons. For instance one might assess probability with 50% price increase, whereas another person may assess it with a 40% price increase.

  Again, if one believes that price will increase by 50%, then to be consistent, it is not possible to believe that price will go down by 50%

Fits well with Bayesian Statisties

Axiomatic :

- This is the unifying perspective, so that it can be used with the other three approaches.

Axiomatic perspective of probability measure suggests that

probability is a function that maps an event to a number, satisfying a set of criteria often termed as

Axioms of probability